ENSLAPP-L-493/94
hep-th/9410203
revised version
SWAT-95-77

Regular Conjugacy Classes in the Weyl Group and Integrable Hierarchies

F. Delduc and L. Fehér*

Laboratoire de Physique Théorique†
ENS Lyon, 46 allée d'Italie
F-69364 Lyon Cedex 07, France

Abstract. Generalized KdV hierarchies associated by Drinfeld-Sokolov reduction to grade one regular semisimple elements from non-equivalent Heisenberg subalgebras of a loop algebra $\mathcal{G}\otimes\mathbf{C}[\lambda,\lambda^{-1}]$ are studied. The graded Heisenberg subalgebras containing such elements are labelled by the regular conjugacy classes in the Weyl group $\mathbf{W}(\mathcal{G})$ of the simple Lie algebra $\mathcal{G}$. A representative $w\in\mathbf{W}(\mathcal{G})$ of a regular conjugacy class can be lifted to an inner automorphism of $\mathcal{G}$ given by $\hat{w}=\exp\left(2i\pi\mathrm{adI}_0/m\right)$, where I_0 is the defining vector of an sl_2 subalgebra of $\mathcal{G}$. The grading is then defined by the operator $d_{m,I_0}=m\lambda\frac{d}{d\lambda}+\mathrm{ad}I_0$ and any grade one regular element Λ from the Heisenberg subalgebra associated to $[w]$ takes the form $\Lambda=(C_++\lambda C_-)$, where $[I_0,C_-]=-(m-1)C_-$ and C_+ is included in an sl_2 subalgebra containing I_0. The largest eigenvalue of $\mathrm{ad}I_0$ is $(m-1)$ except for some cases in F_4, $E_{6,7,8}$. We explain how these Lie algebraic results follow from known results and apply them to construct integrable systems. If the largest $\mathrm{ad}I_0$ eigenvalue is $(m-1)$, then using any grade one regular element from the Heisenberg subalgebra associated to $[w]$ we can construct a KdV system possessing the standard $\mathcal{W}$-algebra defined by I_0 as its second Poisson bracket algebra. For $\mathcal{G}$ a classical Lie algebra, we derive pseudo-differential Lax operators for those non-principal KdV systems that can be obtained as discrete reductions of KdV systems related to gl_n. Non-abelian Toda systems are also considered.

*Present address: Department of Physics, University of Swansea, Singleton Park, Swansea SA2 8PP, UK. On leave from Theoretical Physics Department of Szeged University, H-6720 Szeged, Hungary.
†ENSLAPP, URA 14-36 du CNRS, associé à l'E.N.S. de Lyon et au L.A.P.P.

Contents

1 Introduction

The purpose of this paper is to contribute to the classification of generalized KdV systems that may be obtained from the Drinfeld-Sokolov approach to integrable hierarchies. One of the main achievements presented in the seminal paper [1] by Drinfeld and Sokolov was the interpretation in terms of affine Lie algebras of the n-KdV hierarchies defined by Gelfand and Dickey in [2, 3] and Adler in [4] in terms of the calculus of pseudo-differential operators. The phase space consisting of scalar Lax operators

$$L = \partial^n + u_1\partial^{n-1} + \cdots + u_{n-1}\partial + u_n, \qquad u_i \in C^\infty(S^1, \mathbf{C}), \tag{1.1}$$

was interpreted as the reduced phase space following a Hamiltonian symmetry reduction applied to the dual of an affine Lie algebra. This explained the origin of the quadratic Adler-Gelfand-Dickey Poisson bracket as a reduced Lie-Poisson bracket and also explained the commuting Hamiltonians generated by residues of fractional powers of L as being reductions of those obtained by applying the Adler-Kostant-Symes scheme to the affine Lie algebra (see also [5]). The properties of the matrix

$$\Lambda_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ \lambda & 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{1.2}$$

played a crucial role in the construction. The centralizer of Λ_n in the loop algebra $\ell(gl_n) := gl_n \otimes \mathbf{C}[\lambda, \lambda^{-1}]$ is a graded maximal abelian subalgebra, which becomes the principal Heisenberg subalgebra upon central extension [6]. The commuting flows were constructed out of this abelian subalgebra making essential use of the principal grading and the regularity of the element Λ_n that has grade one. The other main achievement of Drinfeld and Sokolov was the derivation of new KdV type hierarchies by generalizing the construction to an arbitrary affine Lie algebra using the respective principal Heisenberg subalgebra and its grade one regular element. Like the KdV type systems of [1], the affine Toda systems are also based on the principal Heisenberg subalgebra, with the grading and the regular element of grade one playing an important role.

The generalized KdV systems that will be studied in this paper will be associated to regular elements of grade one from certain non-principal Heisenberg subalgebras of $\ell(\mathcal{G}) := \mathcal{G} \otimes [\lambda, \lambda^{-1}]$ for $\mathcal{G}$ a simple Lie algebra using the Hamiltonian reduction technique of [1]. Related non-abelian affine Toda systems will be also presented.

Generalizations of the Drinfeld-Sokolov construction of integrable hierarchies have already been considered in the literature. Soon after [1], Wilson [7] suggested associating systems of modified KdV and Toda type to any grade one semisimple element of any affine Lie algebra, with respect to a grading defined by an automorphism of finite order of the corresponding finite dimensional simple Lie algebra. In the context of Toda field theories, similar proposals can be found in [8, 9, 10]. Concerning the important, apparently still open, problem of classifying the gradings that admit a grade one semisimple element, some progress was made in [11, 10]. The construction of systems of modified KdV type can be done without any reference to a gauge freedom, while the presence of a non-trivial gauge freedom is a crucial ingredient in

the construction of the KdV type systems in [1]. In the unpublished work [11], the reduction procedure of [1] was generalized in order to obtain generalized Miura maps for associating KdV type systems to those of modified KdV type. It was also realized in [11] that the semisimple element and the gradings involved in the generalized Drinfeld-Sokolov reduction must satisfy a certain non-degeneracy condition, which is required for the existence of the global, polynomial gauges that define the KdV fields as in [1]. More recently, the ideas of [7] were resurrected and made concrete by de Groot et al [12, 13, 14, 15] taking advantage of the theory of non-equivalent graded Heisenberg subalgebras in the affine Lie algebras developed by Kac and Peterson [16]. In [12] it was suggested to use any graded element Λ with non-zero grade from any Heisenberg subalgebra of an affine Lie algebra in a generalized Drinfeld-Sokolov reduction procedure. Such an element Λ is automatically semisimple and in [12] two types of systems, called type I and type II, were distinguished according to whether Λ is regular or non-regular. The notion of regularity is defined below. In the type I cases it is possible to verify the existence of the polynomial gauges ("DS gauges") required for the construction of KdV type systems. This in general is not so in the type II cases and has to be imposed as an extra condition for obtaining KdV type systems.

In fact the approach used in [12] is almost the same as the one in [11]. In the setup of [12] the semisimple element Λ can have any non-zero grade, but in the most interesting cases when Λ has grade one the two methods almost always coincide. Indeed in the case of the classical simple Lie algebras we are aware of no exceptions. An advantage of the approach used in [12] is that it incorporates a universal definition of the gauge group which is applicable to any graded semisimple element Λ and implies the existence of polynomial gauge fixings if Λ is regular.

According to the above, one can associate generalized KdV systems to certain graded semisimple elements of the affine Lie algebras that include the regular elements of minimal non-zero (say positive) grade taken from the non-equivalent graded Heisenberg subalgebras. It appears a reasonable strategy to first explore the systems that may be associated to the non-equivalent regular semisimple elements of minimal grade. Progress in this direction was reported in [17, 18], where the case of the affine Lie algebra $\ell(gl_n)$ was considered. In this case the graded Heisenberg subalgebras are classified by the partitions on n [16, 19] and it was verified in [17] that only the partitions of n into sums of equal numbers, $n = sp$, and into sums of equal numbers plus one, $n = sp + 1$, admit a graded regular element. A generalized Drinfeld-Sokolov reduction based on a grade one regular element from the Heisenberg subalgebra associated to the partition $n = sp$ was analyzed in [17] and was found to lead to the matrix version of the Gelfand-Dickey hierarchy given by Lax operators of the form

$$L = Q\partial^p + u_1\partial^{p-1} + \cdots + u_{p-1}\partial + u_p, \qquad u_i \in C^\infty(S^1, gl_s), \tag{1.3}$$

where Q is a diagonal constant matrix with distinct, non-zero entries. In the case $n = sp+1$ the analogous Drinfeld-Sokolov reduction (see [18]) yields a hierarchy associated to a more exotic looking $s \times s$ matrix Lax operator:

$$L = Q\partial^p + u_1\partial^{p-1} + \cdots + u_{p-1}\partial + u_p - y_+(\partial + w)^{-1}y_-^t\,, \tag{1.4}$$

where the fields u_i vary like in (1.3), $y_\pm \in C^\infty(S^1, \mathbf{C}^s)$ and $w \in C^\infty(S^1, \mathbf{C})$. For the history of this model and for related recent developments on KdV type hierarchies, the reader may consult refs. [20, 21, 22, 23, 24, 25], in all of which different methods to those in [17, 18] were used.

In none of the above mentioned papers had it been realized that a classification of the graded regular semisimple elements of the affine Lie algebras can be extracted from known results. We now explain this in the non-twisted case. Let $\mathcal{G}$ be a complex simple Lie algebra. Disregarding the central extension, recall from [16] that the graded Heisenberg subalgebras of the non-twisted loop algebra $\ell(\mathcal{G})$ are classified by the conjugacy classes (see [26]) in the Weyl group $\mathbf{W}(\mathcal{G})$ of $\mathcal{G}$. It is also clear from the construction in [16] that the graded regular elements in a Heisenberg subalgebra, $\tilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$ associated to the conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$, correspond to the regular eigenvectors of the Weyl transformation $w \in [w]$ acting on the Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$. In [27] the conjugacy classes in the Weyl group whose representatives admit a regular eigenvector (an eigenvector whose centralizer in $\mathcal{G}$ is $\mathcal{H}$) are themselves called regular. The regular conjugacy classes in the Weyl groups were then all classified by Springer [27]. This yields a classification of the graded regular semisimple elements of $\ell(\mathcal{G})$, since every such element is contained in a graded Heisenberg subalgebra. Although this classification is not yet complete since there are ambiguities in choosing the grading of $\ell(\mathcal{G})$ associated to a conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$, because the construction involves lifting a representative $w \in [w]$ to a finite order inner automorphism $\hat{w} = \exp(2i\pi \mathrm{ad} X)$ of $\mathcal{G}$, we shall see that there exists a natural choice for every regular conjugacy class.

In this paper the above classification of the graded regular semisimple elements of the loop algebras $\ell(\mathcal{G})$ will be developed and applications will be considered concentrating on the classical simple Lie algebras. In addition to the theory of integrable systems, our work is also motivated by the relations between integrable hierarchies and various other subjects of two dimensional theoretical physics, $\mathcal{W}$-algebras and 2d gravity models being prime examples (e.g. [28, 29, 30, 31, 32, 33]). An important question for us is to clarify the relationship between generalized KdV hierarchies and $\mathcal{W}$-algebras, which is well-known in the original Drinfeld-Sokolov case. We will be able to associate a KdV type hierarchy to every grade one regular element from a graded Heisenberg subalgebra of $\ell(\mathcal{G})$ in such a way that the second Poisson bracket of the hierarchy gives a classical $\mathcal{W}$-algebra associated to a corresponding sl_2 subalgebra of $\mathcal{G}$. The set of $\mathcal{W}$-algebras arising in this way is a small subset of the standard $\mathcal{W}$-algebras associated to arbitrary sl_2 embeddings [30, 31]. Our result on the $\mathcal{W}$-algebra structures corresponding to the KdV systems is consistent with the results in [15], where a $\mathcal{W}$-subalgebra was exhibited in the second Poisson bracket algebra for a certain class of generalized KdV hierarchies. By the method of [12, 13], these hierarchies are associated to a graded semisimple element Λ subject to a certain non-degeneracy condition, which is satisfied in all the cases that we shall consider.

Before describing the content of the paper in more detail, it is worthwhile to recapitulate the essence of the use of a graded regular semisimple element of non-zero grade to integrable systems in technical terms. An element Λ of a non-twisted loop algebra $\ell(\mathcal{G})$, where $\mathcal{G}$ is a simple Lie algebra or gl_n, is called *semisimple* if it defines a direct sum decomposition

$$\ell(\mathcal{G}) = \mathrm{Ker}(\mathrm{ad}\,\Lambda) + \mathrm{Im}(\mathrm{ad}\,\Lambda). \tag{1.5}$$

By definition, a semisimple element Λ is *regular* if $\mathrm{Ker}(\mathrm{ad}\,\Lambda) \subset \ell(\mathcal{G})$ is an *abelian* subalgebra. The $\mathbf{Z}$-grading in which Λ is supposed to be homogeneous with non-zero grade is defined by the eigenspaces of a linear operator $d_{N,Y} : \ell(\mathcal{G}) \to \ell(\mathcal{G})$,

$$d_{N,Y} = N\lambda \frac{d}{d\lambda} + \mathrm{ad}\, Y, \tag{1.6}$$

where N is a non-zero integer and $Y \in \mathcal{G}$ is diagonalizable with integer eigenvalues in the adjoint representation. If one has such an element, then $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$ is a *graded, maximal* abelian subalgebra. Note also that $\mathrm{ad}\,Y$ defines a grading $\mathcal{G} = \oplus_i \mathcal{G}_i$ of $\mathcal{G}$. The most important graded regular semisimple elements are of small grade taking the form

$$\Lambda = C_+ + \lambda C_- \qquad \text{with some} \qquad C_\pm \in \mathcal{G}. \tag{1.7}$$

The integrable hierarchies of our interest are given by Hamiltonian flows on a phase space consisting of first order differential operators $\mathcal{L}$ of the type

$$\mathcal{L} = \partial + j + \Lambda \qquad \text{with} \quad j : S^1 \to \sum_{i<k} \ell(\mathcal{G})_i, \tag{1.8}$$

where $\ell(\mathcal{G})_i \subset \ell(\mathcal{G})$ is the grade i eigensubspace of $d_{N,Y}$ and $k > 0$ is the grade of Λ. In addition to being restricted to grades strictly smaller than the grade of the leading term Λ, the field j in (1.8) is usually also subject to further constraints (e.g. it often varies in $\mathcal{G} \subset \ell(\mathcal{G})$ only) and to a gauge freedom specific to the system. Since the field j is periodic (being a function on the space S^1), one can consider the monodromy matrix of $\mathcal{L}$. The point is that under the above assumptions one may obtain commuting *local* Hamiltonians from the monodromy invariants determined by the "abelianization" of $\mathcal{L}$ [1, 7, 9, 11, 12]. This abelianization is essentially a perturbative diagonalization which is achieved by transforming $\mathcal{L}$ (1.8) according to

$$(\partial + j + \Lambda) \mapsto e^{\mathrm{ad}\,F}\,(\partial + j + \Lambda) := (\partial + h + \Lambda), \tag{1.9}$$

where F and h are infinite series required to take their values in appropriate graded subspaces in the decomposition (1.5):

$$F : S^1 \to (\mathrm{Im}(\mathrm{ad}\,\Lambda))_{<0}\,, \qquad h : S^1 \to (\mathrm{Ker}(\mathrm{ad}\,\Lambda))_{<k}\,. \tag{1.10}$$

In fact, the above assumptions ensure that (1.9), (1.10) can be solved recursively, grade by grade, for both $F(j)$ and $h(j)$ and the solution is given by unique *differential polynomials* in the components of j. The local monodromy invariants are the integrals over S^1 of the graded components of the resulting $h(j)$. In an appropriate hamiltonian setting, these provide the Hamiltonians that generate a hierarchy of commuting evolution equations.

The rest of this paper is organized as follows. Sections 2, 3 and 4 are devoted to presenting some Lie algebraic results relevant for the classification of generalized KdV systems. In Section 2 it is explained that the classification of the graded regular semisimple elements of a loop algebra $\ell(\mathcal{G})$ can be reduced to the classification of the regular eigenvectors of representatives of the conjugacy classes in the Weyl group $\mathbf{W}(\mathcal{G})$ of $\mathcal{G}$ thanks to results in [16]. The solution of this classification problem which is due to Springer [27], is summarized in Tables 1, 2 and 3 in Section 3 for $\mathcal{G}$ a classical simple Lie algebra.

In Section 4 we describe a connection between the regular conjugacy classes in $\mathbf{W}(\mathcal{G})$, with associated grade one regular semisimple elements in $\ell(\mathcal{G})$, and certain sl_2 subalgebras in the classical Lie algebra $\mathcal{G}$. For every regular conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$ of order m, we shall exhibit a lift $\hat{w}$ of a representative $w \in [w]$ having the form

$$\hat{w} = \exp\left(2i\pi \mathrm{ad} I_0/m\right), \tag{1.11}$$

where I_0 is the defining vector [34] of an sl_2 subalgebra of $\mathcal{G}$ and the largest eigenvalue of $\mathrm{ad}I_0$ is $(m-1)$. The order of the inner automorphism $\hat{w}$ of $\mathcal{G}$ is νm, where ν is 1 or 2 depending on whether $\mathrm{ad}I_0$ has only integral or also half-integral eingenvalues. Actually $\nu = 1$ in almost all cases. Using this $\hat{w}$ in the Kac-Peterson construction of the graded Heisenberg subalgebra, $\tilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$ associated to $[w] \subset \mathbf{W}(\mathcal{G})$, induces the $\mathbf{Z}/\nu$ grading on $\ell(\mathcal{G})$ defined by the operator $d_{m,I_0} = m\lambda\frac{d}{d\lambda} + \mathrm{ad}I_0$. This is the natural grading of $\ell(\mathcal{G})$ which we associate to $[w]$. We then show that every graded regular element $\Lambda \in \tilde{\mathcal{H}}_{\hat{w}}$ of minimal positive grade, in fact d_{m,I_0} grade one, has the form (1.7), *where C_+ can be included in an sl_2 subalgebra containing also I_0*. That is there exists $I_- \in \mathcal{G}$ for which $[I_0, I_\pm] = \pm I_\pm$, $[I_+, I_-] = 2I_0$ holds with $I_+ := C_+$ contained in $\Lambda = (C_+ + \lambda C_-)$.

The above connection between regular conjugacy classes in $\mathbf{W}(\mathcal{G})$ and sl_2 subalgebras in $\mathcal{G}$ generalizes and in many cases is implied by the classical result of Kostant [35] on the connection between the Coxeter class in $\mathbf{W}(\mathcal{G})$ and the principal sl_2 subalgebra in $\mathcal{G}$. In the main text we shall take $\mathcal{G}$ to be a classical Lie algebra, but in Appendix A we discuss the connection between regular conjugacy classes in $\mathbf{W}(\mathcal{G})$ and sl_2 embeddings in $\mathcal{G}$ for an arbitrary simple Lie algebra too. In the algebras F_4 and $E_{6,7,8}$ we find that $(m-1)$ in (1.11) is smaller than the largest $\mathrm{ad}I_0$ eigenvalue in some cases, but equality holds for every *regular primitive* conjugacy class. As will be clear from our references, we do not claim credit for original group theoretic results. However, by inspecting and systematizing a number of isolated results, we will be able to formulate and verify interesting general statements, which are worth knowing but to our knowledge are not available in the literature.

We turn to the application of the above results to the construction of KdV type integrable hierarchies in Section 5. In Subsection 5.1 we associate a KdV type system to every grade one regular semisimple element $\Lambda \in \tilde{\mathcal{H}}_{\hat{w}}$. This hierarchy will be obtained by a direct generalization of the standard Drinfeld-Sokolov reduction. We assume that the largest eigenvalue of $\mathrm{ad}I_0$ equals $(m-1)$ in (1.11), which is always satisfied if $\mathcal{G}$ is a classical simple Lie algebra or G_2. The second Poisson bracket algebra of the resulting generalized KdV hierarchy is then the $\mathcal{W}$-algebra [30, 31] belonging to the sl_2 embedding defined by I_0. In Subsection 5.2 we derive Gelfand-Dickey type Lax operators for a subset of the generalized KdV systems. These systems correspond to conjugacy classes in the Weyl group of a classical Lie algebra given by the product of Coxeter elements in a regular subalgebra composed of A and C type simple factors. They turn out to be "discrete reductions" of generalized KdV systems related to gl_n given by Lax operators of the form in (1.3) and (1.4). In Section 6 we briefly comment on non-abelian affine Toda systems and present the detailed form of the non-abelian affine Toda equation corresponding to the regular, primitive (semi-Coxeter) conjugacy class $(\bar{p}, \bar{p}) \subset \mathbf{W}(D_{2p})$.

Finally, we give our conclusions and comment on some open problems in Section 7.

2 Heisenberg subalgebras and the Weyl group

Let $\mathcal{G}$ be a complex simple Lie algebra. Consider the Lie algebra $\ell(\mathcal{G})$ of Laurent polynomials, $\ell(\mathcal{G}) := \mathcal{G} \otimes \mathbf{C}[\lambda, \lambda^{-1}]$, in the spectral parameter λ. For any graded regular semisimple element $\Lambda \in \ell(\mathcal{G})$, $\mathrm{Ker}(\mathrm{ad}\,\Lambda) \subset \ell(\mathcal{G})$ is a graded maximal abelian subalgebra, which becomes a Heisenberg subalgebra upon centrally extending $\ell(\mathcal{G})$. In order to find the graded regular semisimple elements of $\ell(\mathcal{G})$, it is therefore enough to inspect the maximal abelian subalgebras of $\ell(\mathcal{G})$ that underlie the graded Heisenberg subalgebras of the central extension $\widehat{\mathcal{G}}$ of $\ell(\mathcal{G})$, and select those which contain graded regular elements. With respect to the adjoint action of an appropriate group associated to $\ell(\mathcal{G})$, the non-equivalent graded Heisenberg subalgebras of $\widehat{\mathcal{G}}$ are *classified* by the conjugacy classes in the Weyl group of $\mathcal{G}$ [16]. See also [36, 37] for the precise statement. Next we recall the main points of the construction on which this classification is based. Note that, by disregarding the central extension, a maximal abelian subalgebra of $\ell(\mathcal{G})$ will be often referred to as a Heisenberg subalgebra throughout the text.

Suppose that $\mathcal{H} \subset \mathcal{G}$ is a Cartan subalgebra and τ is a *finite order, inner* automorphism of $\mathcal{G}$ that normalizes $\mathcal{H}$. Consider the following models of $\ell(\mathcal{G})$ and its twisted realization $\ell(\mathcal{G}, \tau)$:

$$\begin{aligned} \ell(\mathcal{G}) &= \{ F \,|\, F : \mathbf{R} \to \mathcal{G}, \quad F(\theta + 2\pi) = F(\theta) \}, \\ \ell(\mathcal{G}, \tau) &= \{ f \,|\, f : \mathbf{R} \to \mathcal{G}, \quad f(\theta + 2\pi) = \tau\left(f(\theta)\right) \}. \end{aligned} \tag{2.1}$$

Since τ is inner, $\ell(\mathcal{G})$ and $\ell(\mathcal{G}, \tau)$ are isomorphic [6, 38]. To see this one writes τ as

$$\tau = e^{2i\pi \mathrm{ad} X}, \quad X = Y/N, \tag{2.2}$$

where N is the order of τ, $\tau^N = \mathrm{id}$, and $Y \in \mathcal{G}$ is diagonalizable. The choice of Y is not unique. The isomorphism $\eta : \ell(\mathcal{G}, \tau) \to \ell(\mathcal{G})$ is given by "untwisting" as follows:

$$\eta : f \mapsto F\,, \qquad F(\theta) := e^{-i\theta \mathrm{ad}\, X}\left(f(\theta)\right). \tag{2.3}$$

The "twisted homogeneous Heisenberg subalgebra" $\ell(\mathcal{H}, \tau)$,

$$\ell(\mathcal{H}, \tau) = \{ f \,|\, f : \mathbf{R} \to \mathcal{H}, \quad f(\theta + 2\pi) = \tau\left(f(\theta)\right) \}, \tag{2.4}$$

is a maximal abelian subalgebra of $\ell(\mathcal{G}, \tau)$. The image $\tilde{\mathcal{H}}_\tau := \eta[\ell(\mathcal{H}, \tau)]$ of the twisted homogeneous Heisenberg subalgebra is a maximal abelian subalgebra of $\ell(\mathcal{G})$. The natural grading on $\ell(\mathcal{G}, \tau)$ is the homogeneous grading defined by the eigensubspaces of $d : \ell(\mathcal{G}, \tau) \to \ell(\mathcal{G}, \tau)$,

$$d := -iN \frac{d}{d\theta}. \tag{2.5}$$

The isomorphism η induces a corresponding grading operator $d_{N,Y} : \ell(\mathcal{G}) \to \ell(\mathcal{G})$,

$$d_{N,Y} := \eta \circ d \circ \eta^{-1} = N\lambda \frac{d}{d\lambda} + \mathrm{ad}\, Y\,, \tag{2.6}$$

where we used the definition $\lambda := e^{i\theta}$. The maximal abelian subalgebras $\ell(\mathcal{H}, \tau) \subset \ell(\mathcal{G}, \tau)$ and $\tilde{\mathcal{H}}_\tau \subset \ell(\mathcal{G})$ are of course graded.

Recall (e.g. [38]) that Weyl group $\mathbf{W}(\mathcal{G})$ of $\mathcal{G}$ may be identified as the group of inner automorphisms of $\mathcal{G}$ that normalize $\mathcal{H}$ modulo the inner automorphisms centralizing $\mathcal{H}$. It is also well-known that any $w \in \mathbf{W}(\mathcal{G})$ may be, in general non-uniquely, lifted to a *finite order* inner automorphism $\hat{w}$ of $\mathcal{G}$ which reduces to w on $\mathcal{H}$, $\hat{w}|_{\mathcal{H}} = w$. It follows that one can associate a graded maximal abelian subalgebra, $\widetilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$, to any element $w \in \mathbf{W}(\mathcal{G})$. To construct $\widetilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$, one first lifts $w \in [w]$ and then performs the above construction using $\hat{w}$ in place of τ in (2.1)–(2.6). Despite the ambiguities involved, it can be shown [16, 36, 37] that conjugate elements of $\mathbf{W}(\mathcal{G})$ give rise to equivalent graded Heisenberg subalgebras and the non-equivalent ones are classified by the conjugacy classes in $\mathbf{W}(\mathcal{G})$.

We now need to construct a graded basis of $\ell(\mathcal{G}, \hat{w})$. This is done as follows. The eigenvalues of $\hat{w}$ on $\mathcal{G}$ are of the form $\omega^{\bar{k}}$ with

$$\omega := \exp\left(2i\pi/N\right) \qquad \text{and} \qquad \bar{k} \in \{0, 1, \ldots, (N-1)\}, \tag{2.7}$$

where N is the order of $\hat{w}$. A basis of $\mathcal{G}$ consisting of eigenvectors of $\hat{w}$ may be given in the form $\{H_{\bar{k},q_{\bar{k}}}\} \cup \{R_{\bar{k},r_{\bar{k}}}\}$ with

$$w(H_{\bar{k},q_{\bar{k}}}) = \omega^{\bar{k}} H_{\bar{k},q_{\bar{k}}}, \quad H_{\bar{k},q_{\bar{k}}} \in \mathcal{H} \qquad \text{and} \qquad \hat{w}(R_{\bar{k},r_{\bar{k}}}) = \omega^{\bar{k}} R_{\bar{k},r_{\bar{k}}}\,, \quad R_{\bar{k},r_{\bar{k}}} \in \mathcal{H}^{\perp}, \tag{2.8}$$

that is by separately diagonalizing $\hat{w}$ on the Cartan subalgebra $\mathcal{H}$ (where it reduces to w) and on its complementary space $\mathcal{H}^{\perp} \subset \mathcal{G}$ spanned by the root vectors. The index $q_{\bar{k}}$, similarly $r_{\bar{k}}$, counts the multiplicity of the corresponding eigenvalue, which can be also zero of course. The desired graded basis of $\ell(\mathcal{G}, \hat{w})$ consists of the elements

$$z^k H_{\bar{k},q_{\bar{k}}} \quad \text{and} \quad z^k R_{\bar{k},r_{\bar{k}}} \quad \text{where} \quad z := \exp\left(i\theta/N\right), \quad k = \bar{k} \bmod N. \tag{2.9}$$

By definition, a graded element $z^k H_{\bar{k},q_{\bar{k}}} \in \ell(\mathcal{H}, \hat{w}) \subset \ell(\mathcal{G}, \hat{w})$ of grade k is *regular* if

$$\ell(\mathcal{G}, \hat{w}) \supset \mathrm{Ker}\left(\mathrm{ad}\, z^k H_{\bar{k},q_{\bar{k}}}\right) = \ell(\mathcal{H}, \hat{w}). \tag{2.10}$$

It is easy to see that (2.10) is equivalent to

$$\mathcal{G} \supset \mathrm{Ker}\left(\mathrm{ad}\, H_{\bar{k},q_{\bar{k}}}\right) = \mathcal{H}. \tag{2.11}$$

Equations (2.10) and (2.11) refer respectively to infinite and finite dimensional Lie algebras. Using standard terminology in the finite dimensional case, $H \in \mathcal{H}$ is by definition *regular* if its centralizer in $\mathcal{G}$ is $\mathcal{H}$. Hence the equivalence of (2.10) and (2.11) means that $z^k H_{\bar{k},q_{\bar{k}}} \in \ell(\mathcal{H}, \hat{w})$ *is a regular semisimple element of* $\ell(\mathcal{G}, \hat{w})$ *if and only if* $H_{\bar{k},q_{\bar{k}}} \in \mathcal{H}$ *is a regular semisimple element of* $\mathcal{G}$. In principle, this simple statement should make it possible to find all graded regular semisimple elements of $\ell(\mathcal{G})$.

In order to find the graded regular semisimple elements of $\ell(\mathcal{G})$, one needs to select the conjugacy classes $[w] \subset \mathbf{W}(\mathcal{G})$ for which the graded maximal abelian subalgebra $\widetilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$ contains a graded regular element. By the isomorphism between $\ell(\mathcal{G}, \hat{w})$ and $\ell(\mathcal{G})$ that brings $\ell(\mathcal{H}, \hat{w})$ into $\widetilde{\mathcal{H}}_{\hat{w}}$ and the statement above, this problem is equivalent to selecting the conjugacy classes in $\mathbf{W}(\mathcal{G})$ whose representatives admit a regular eigenvector. A conjugacy class with this property is called a *regular conjugacy class* in [27], where all such conjugacy classes have been listed.

Remark. It is apparent from the above construction of the Heisenberg subalgebra $\tilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$ associated to $[w] \subset \mathbf{W}(\mathcal{G})$ that the corresponding grading of $\ell(\mathcal{G})$ depends on the choice of the finite order inner automorphism $\hat{w}$ used for defining the lift of a representative $w \in [w]$. As the grading plays a crucial role in the Drinfeld-Sokolov construction, a clarification of this ambiguity, in terms of the classification of finite order automorphisms due to Kac [6, 38], would be desirable. This problem will not be addressed in the present paper. Rather, in Section 4 and in Appendix A, a distinguished lift having the nice properties in (1.11) will be exhibited for every regular conjugacy class in the Weyl group.

3 Regular conjugacy classes in the Weyl group

The conjugacy classes in the Weyl group are described in [26] for all simple Lie algebras, and the regular conjugacy classes (which admit a regular eigenvector) are described in [27]. In this section we recall the relevant results of [27] in the form of tables for the classical simple Lie algebras, which will be used in our applications later. In these tables we shall also present the explicit form of the regular eigenvectors for convenient representatives of the regular conjugacy classes. The eigenvectors are not given in [27], but can be easily computed. As a matter of fact the classification of the regular conjugacy classes can be also derived straightforwardly by explicitly diagonalizing a representative for each conjugacy class and inspecting the eigenvectors. In our study originally we used this "brute force" approach, but after learning the elegant work of Springer [27] this explicit inspection became superfluous and will not be presented apart from some remarks. By means of the natural scalar product, the Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ will be always identified with the space of roots $\mathcal{H}^*$ in this section.

3.1 Regular conjugacy classes in $\mathbf{W}(A_{n-1})$

The Cartan subalgebra of A_{n-1} may be identified with the subspace of the vector space spanned by n orthonormal vectors ϵ_l, $l = 1, \ldots, n$ which is orthogonal to the vector $\sum_{l=1}^n \epsilon_l$. The roots of A_{n-1} are the vectors $\epsilon_l - \epsilon_{l'}$, $l \neq l'$. An element

$$H = \sum_{l=1}^{n} h_l \epsilon_l, \quad \sum_{l=1}^{n} h_l = 0, \tag{3.1}$$

of the Cartan subalgebra is regular if and only if for any two distinct indices l and l', $h_l \neq h_{l'}$. The Weyl group $\mathbf{W}(A_{n-1})$ is the permutation group of the n vectors ϵ_l. The conjugacy classes in $\mathbf{W}(A_{n-1})$ are in one-to-one correspondence with the partitions of n,

$$(n_1, \ldots, n_s), \quad \sum_{k=1}^{s} n_k = n, \tag{3.2}$$

where the n_k ($k = 1, \ldots, s$) are non-increasing positive integers giving the length of the cycles inside a given conjugacy class. To describe the action of a representative w of the conjugacy class associated to the partition (3.2), it is useful to re-label the basis vectors as follows:

$$\epsilon_{k,i_k} := \epsilon_l, \quad l = (\sum_{m=1}^{k-1} n_m) + i_k, \quad k = 1, \ldots, s, \; i_k = 1, \ldots, n_k. \tag{3.3}$$

The action of w on these basis vectors may be chosen to be

$$w(\epsilon_{k,1}) = \epsilon_{k,n_k}, \quad w(\epsilon_{k,i_k}) = \epsilon_{k,i_k-1}, \; i_k \neq 1. \tag{3.4}$$

Since w does not mix vectors corresponding to different cycles, one obtains a basis of eigenvectors by considering each cycle separately. Let us focus our attention on the kth cycle of length n_k, and define $\omega_k := e^{\frac{2i\pi}{n_k}}$. The eigenvalues of w on the space spanned by the vectors ϵ_{k,i_k} $(i_k = 1, \ldots, n_k)$ are $(\omega_k)^{j_k}$, $j_k = 0, \ldots, n_k - 1$, and the corresponding eigenvectors, denoted as $H_{j_k}(k)$, are

$$H_{j_k}(k) = \sum_{i_k=1}^{n_k} (\omega_k)^{(i_k-1)j_k} \epsilon_{k,i_k}. \tag{3.5}$$

One can look for a regular eigenvector of w in the form

$$H = \sum_{k=1}^{s} d_k H_{j_k}(k). \tag{3.6}$$

The eigenvalues of w on those $H_{j_k}(k)$ for which $d_k \neq 0$ must be equal, and $h_l \neq h_{l'}$ must hold for any distinct indices when re-expanding H (3.6) in the form (3.1). These conditions lead to the result summarized in Table 1. Note that $\gcd(p, j)$ denotes the greatest common divisor of p and j, and in the case $j = 0$ ($\gcd(p, 0) = 1$) the condition $\sum d_k = 0$ must be also imposed for the eigenvector to belong to the Cartan subalgebra of A_{n-1}.

Conjugacy class	Eigenvector	Eigenvalue	Regularity conditions
$(p, \ldots, p),\; p \geq 1$	$\sum_{k=1}^{s} d_k H_j(k)$	$\exp(\frac{2i\pi j}{p})$	$\gcd(p, j) = 1$ $(d_k)^p \neq (d_{k'})^p,\; d_k \neq 0$ if $p > 1$
$(p, \ldots, p, 1),\; p > 1$	$\sum_{k=1}^{s-1} d_k H_j(k)$	$\exp(\frac{2i\pi j}{p})$	$\gcd(p, j) = 1$ $(d_k)^p \neq (d_{k'})^p,\; d_k \neq 0$

Table 1: Regular eigenvectors of $w \in \mathbf{W}(A_{n-1})$.
$H_j(k) = \sum_{l=1}^{p} \exp(\frac{2\pi i(l-1)j}{p}) \epsilon_{(k-1)p+l}$ for $0 \leq j \leq (p-1)$.

3.2 Regular conjugacy classes in $\mathbf{W}(D_n)$

The Cartan subalgebra of D_n may be identified with the vector space spanned by n orthonormal vectors ϵ_l, $l = 1, \ldots, n$. The roots of D_n are the vectors $\pm\epsilon_l \pm \epsilon_{l'}$, $l \neq l'$. An element $H = \sum_{l=1}^{n} h_l \epsilon_l$ of the Cartan subalgebra is regular if and only if for any two distinct indices l and l', $h_l \neq \pm h_{l'}$. The Weyl group $\mathbf{W}(D_n)$ consists of the permutations of the vectors ϵ_l and the sign changes of an arbitrary even number of them [26]. A so called "signed partition" of n can be associated to each conjugacy class,

$$(n_1, \ldots, n_r, \bar{n}_{r+1}, \ldots, \bar{n}_s), \quad \sum_{k=1}^{s} n_k = n, \tag{3.7}$$

where $n_1, \ldots, n_r$ (resp. $n_{r+1}, \ldots, n_s$) is a sequence of non-increasing positive integers which are the lengths of the positive (negative) cycles. The number of negative cycles $s - r$ is even. It is

shown in [26] that a unique conjugacy class in $\mathbf{W}(D_n)$ is associated to such a signed partition, except when all cycles are positive of even length, in which case the same partition corresponds to two distinct conjugacy classes. To describe the action of a representative w of the conjugacy class associated to the signed partition (3.7), we follow [39] and introduce the adapted basis vectors ϵ_{k,i_k} ($k = 1, \ldots, s$, $i_k = 1, \ldots, n_k$) similarly to (3.3). The action of w on these basis vectors may be chosen to be:

$$w(\epsilon_{k,1}) = \epsilon_{k,n_k}, \quad w(\epsilon_{k,i_k}) = \epsilon_{k,i_k-1}, \ i_k \neq 1, \quad \text{if} \quad 1 \leq k \leq r, \tag{3.8}$$

and

$$w(\epsilon_{k,1}) = -\epsilon_{k,n_k}, \quad w(\epsilon_{k,i_k}) = \epsilon_{k,i_k-1}, \ i_k \neq 1, \quad \text{if} \quad r < k \leq s. \tag{3.9}$$

In the case of a signed partition with only positive even cycles, a representative w' of the second conjugacy class may be chosen to differ from w (3.8) in the first cycle only, where it contains two sign changes:

$$w'(\epsilon_{1,1}) = -\epsilon_{1,n_1}, \quad w'(\epsilon_{1,2}) = -\epsilon_{1,1}, \quad w'(\epsilon_{1,i_1}) = \epsilon_{1,i_1-1}, \ i_1 \neq 1, 2. \tag{3.10}$$

In fact, the conjugacy class of w' is not regular. If $H_j(k)$ and $\tilde{H}_j(k)$ denote a basis of the eigenvectors of w on the space spanned by $\epsilon_{k,1}, \ldots, \epsilon_{k,n_k}$ for $k = 1, \ldots, r$ and for $k = r+1, \ldots, s$, respectively, then the general eigenvector H takes the form

$$H = \sum_{k=1}^{r} d_k H_{j_k}(k) + \sum_{k=r+1}^{s} d_k \tilde{H}_{j_k}(k), \tag{3.11}$$

where the eigenvalues of w associated to the terms with nonzero d_k must be equal. The eigenvector $H_{j_k}(k)$, with eigenvalue $(\omega_k)^{j_k}$ for $j_k = 0, \ldots, n_k - 1$, is given in (3.5). Introduce the notation $\tilde{\omega}_k := e^{\frac{2i\pi}{2n_k}}$. The eigenvector $\tilde{H}_{j_k}(k)$, with eigenvalue $(\tilde{\omega}_k)^{2j_k-1}$ for $j_k = 1, \ldots, n_k$, is defined by

$$\tilde{H}_j(k) = \sum_{i_k=1}^{n_k} (\tilde{\omega}_k)^{(i_k-1)(2j_k-1)} \epsilon_{k,i_k}. \tag{3.12}$$

As can be verified by inspecting formula (3.11), the regular conjugacy classes [27] and the corresponding regular eigenvector are the ones given in Table 2, where q is an integer.

Conjugacy class	Eigenvector	Eigenvalue	Regularity conditions
$(p, \ldots, p)$ $p = 2q+1, \ q \geq 0$	$\sum_{k=1}^{s} d_k H_j(k)$	$\exp(\frac{2i\pi j}{p})$	$\gcd(p, j) = 1$ $(d_k)^p \neq \pm(d_{k'})^p, \ d_k \neq 0$ if $p > 1$
$(p, \ldots, p, 1)$ $p = 2q+1, \ q > 0$	$\sum_{k=1}^{s-1} d_k H_j(k)$	$\exp(\frac{2i\pi j}{p})$	$\gcd(p, j) = 1$ $(d_k)^p \neq \pm(d_{k'})^p, \ d_k \neq 0$
$(\bar{p}, \ldots, \bar{p})$ $p \geq 1, \ s = 2q, \ q \geq 1$	$\sum_{k=1}^{s} d_k \tilde{H}_j(k)$	$\exp(\frac{2i\pi(2j-1)}{2p})$	$\gcd(p, 2j-1) = 1$ $(d_k)^p \neq \pm(d_{k'})^p, \ d_k \neq 0$ if $p > 1$
$(\bar{p}, \ldots, \bar{p}, 1)$ $p \geq 1, \ s = 2q+1, \ q \geq 1$	$\sum_{k=1}^{s-1} d_k \tilde{H}_j(k)$	$\exp(\frac{2i\pi(2j-1)}{2p})$	$\gcd(p, 2j-1) = 1$ $(d_k)^p \neq \pm(d_{k'})^p, \ d_k \neq 0$
$(\bar{p}, \ldots, \bar{p}, \bar{1})$ $p > 1, \ s = 2q, \ q \geq 1$	$\sum_{k=1}^{s-1} d_k \tilde{H}_j(k)$	$\exp(\frac{2i\pi(2j-1)}{2p})$	$\gcd(p, 2j-1) = 1$ $(d_k)^p \neq \pm(d_{k'})^p, \ d_k \neq 0$

Table 2: Regular eigenvectors of $w \in \mathbf{W}(D_n)$.
$H_j(k) = \sum_{l=1}^{p} \exp(\frac{2\pi i(l-1)j}{p})\epsilon_{(k-1)p+l}$ for $0 \leq j \leq (p-1)$.
$\tilde{H}_j(k) = \sum_{l=1}^{p} \exp(\frac{\pi i(l-1)(2j-1)}{p})\epsilon_{(k-1)p+l}$ for $1 \leq j \leq p$.

3.3 Regular conjugacy classes in $\mathbf{W}(B_n) \simeq \mathbf{W}(C_n)$

We identify the Cartan subalgebra of B_n or C_n with the vector space spanned by n orthonormal vectors ϵ_l, $l = 1, \ldots, n$. The roots of B_n are $\pm\epsilon_l \pm \epsilon_{l'}$, $l \neq l'$ and $\pm\epsilon_l$. Those of C_n are $\pm\epsilon_l \pm \epsilon_{l'}$, $l \neq l'$ and $\pm 2\epsilon_l$. Thus an element $H = \sum_{l=1}^{n} h_l \epsilon_l$ of the Cartan subalgebra is regular if and only if for any two distinct indices l and l', $h_l \neq \pm h_{l'}$ and for any l, $h_l \neq 0$. The Weyl groups of B_n and C_n are isomorphic, they consist of the permutations of the basis vectors ϵ_l and the sign changes of arbitrary subsets of them. The conjugacy classes of these groups [26] are in one-to-one correspondence with the signed partitions of n:

$$(n_1, \ldots, n_r, \bar{n}_{r+1}, \ldots, \bar{n}_s), \quad \sum_{k=1}^{s} n_k = n, \tag{3.13}$$

where $n_1, \ldots, n_r$ (resp. $n_{r+1}, \ldots, n_s$) is a sequence of non-increasing positive integers which are the lengths of the positive (negative) cycles. The only difference from the D_n case is that there is now no limitation on the number of negative cycles. A representative w of the conjugacy class labelled by the signed partition (3.13) is obtained using the same formulas (3.8), (3.9) as in the D_n case. The supplementary requirement that for any l, $h_l \neq 0$, simply prohibits the appearance of a cycle of length one not contributing to the eigenvector H in (3.11). The result is summarized in Table 3, with the same notations as in Table 2.

Conjugacy class	Eigenvector	Eigenvalue	Regularity conditions
$(p, \ldots, p)$ $p = 2q+1,\ q \geq 0$	$\sum_{k=1}^{s} d_k H_j(k)$	$\exp(\frac{2i\pi j}{p})$	$\gcd(p, j) = 1$ $(d_k)^p \neq \pm(d_{k'})^p,\ d_k \neq 0$
$(\bar{p}, \ldots, \bar{p}),\ p \geq 1$	$\sum_{k=1}^{s} d_k \tilde{H}_j(k)$	$\exp(\frac{2i\pi(2j-1)}{2p})$	$\gcd(p, 2j-1) = 1$ $(d_k)^p \neq \pm(d_{k'})^p,\ d_k \neq 0$

Table 3: Regular eigenvectors of $w \in \mathbf{W}(B_n) \simeq \mathbf{W}(C_n)$.
$H_j(k) = \sum_{l=1}^{p} \exp(\frac{2\pi i(l-1)j}{p})\epsilon_{(k-1)p+l}$ for $0 \leq j \leq (p-1)$.
$\tilde{H}_j(k) = \sum_{l=1}^{p} \exp(\frac{\pi i(l-1)(2j-1)}{p})\epsilon_{(k-1)p+l}$ for $1 \leq j \leq p$.

The regular conjugacy classes in the Weyl group of an exceptional simple Lie algebra, and in the group obtained as the extension of the Weyl group by the automorphisms of the Dynkin diagram, are also listed in [27]. The classification of regular conjugacy classes in the extended Weyl groups can be used to find graded regular semisimple elements in the twisted affine Lie algebras, similarly to the role of the Weyl group in the non-twisted case to which our attention is restricted in this paper.

4 Heisenberg subalgebras with graded regular elements and sl_2 embeddings

In Section 2 we have seen that the graded Heisenberg subalgebras of the non-twisted loop algebra $\ell(\mathcal{G})$ are classified by the conjugacy classes $[w]$ in $\mathbf{W}(\mathcal{G})$, and the graded regular elements in the Heisenberg subalgebra $\tilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$ arise from the regular eigenvectors of $w \in \mathbf{W}(\mathcal{G})$. For $\mathcal{G}$ a classical Lie algebra, the conjugacy classes in $\mathbf{W}(\mathcal{G})$ listed in the tables of Section 3 parametrize those Heisenberg subalgebras that contain graded regular elements. In this section we describe a relationship between these Heisenberg subalgebras and certain sl_2 subalgebras of $\mathcal{G}$. This relationship consists of two points. First, in the cases when $\tilde{\mathcal{H}}_{\hat{w}}$ contains a graded regular element, the grading $d_{N,Y}$ of $\ell(\mathcal{G})$ induced using the appropriately lifted Weyl group element $\hat{w}$ in the construction of Section 2 takes the form

$$d_{N,Y} = \nu d_{m,I_0}, \quad d_{m,I_0} = m\lambda \frac{d}{d\lambda} + \mathrm{ad} I_0, \tag{4.1}$$

where $I_0 \in \mathcal{G}$ *is the semisimple element of an* sl_2 *subalgebra* $\{I_-, I_0, I_+\} \subset \mathcal{G}$ in the normalization

$$[I_0, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_0. \tag{4.2}$$

Here $\nu = 1$ or 2 depending on whether I_0 determines an integral (even) or a half-integral sl_2 subalgebra of $\mathcal{G}$, and $(m-1)$ is the largest eigenvalue of $\mathrm{ad} I_0$ on $\mathcal{G}$. Second, for any graded regular element $\Lambda \in \tilde{\mathcal{H}}_{\hat{w}}$ of *minimal* positive grade, which has the form

$$\Lambda = C_+ + \lambda C_- \quad \text{with some} \quad C_\pm \in \mathcal{G}, \tag{4.3}$$

we show that C_+ *is the raising element of an* sl_2 *subalgebra containing* I_0. That is there exists $I_- \in \mathcal{G}$ such that (4.2) holds with $I_+ := C_+$. The d_{m,I_0} grade of Λ is one. These statements provide a generalization of the well-known relationship between the principal Heisenberg subalgebra and the principal sl_2 embedding, which underlies the $\mathcal{W}$-algebra structure of the KdV type hierarchies of Drinfeld and Sokolov [1]. In Subsection 4.1 we present a convenient method for constructing explicit realizations of the Heisenberg subalgebras, which will be used to verify the above statements in Subsection 4.2.

It should be emphasized that the above statements refer to a particular lift $\hat{w}$ of $w \in [w]$. A construction of the appropriate lift will be given for any regular conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$.

The correspondence between certain sl_2 subalgebras in $\mathcal{G}$ and certain conjugacy classes in $\mathbf{W}(\mathcal{G})$ has been investigated in the mathematics literature from various viewpoints. The connection of the above mentioned statements to related results in [35, 27, 40] will be explained in Subsection 4.2. See also Appendix A.

4.1 A practical algorithm to construct Heisenberg subalgebras

Recall that the principal Heisenberg subalgebra of $\ell(\mathcal{G})$ is associated to the conjugacy class in $\mathbf{W}(\mathcal{G})$ consisting of Coxeter elements [6]. The Coxeter class is one of the so called *primitive* conjugacy classes of $\mathbf{W}(\mathcal{G})$, which are characterized in [16, 41] by the condition that $\det(1-w) = \det(\mathcal{A})$ for a representative w, where $\mathcal{A}$ is the Cartan matrix of $\mathcal{G}$. In [40] the term "semi-Coxeter" classes is used to denote the primitive conjugacy classes. The most intuitive defining

property of these conjugacy classes is that they do not possess a representative contained in a proper Weyl subgroup of $\mathbf{W}(\mathcal{G})$. The Weyl subgroups of $\mathbf{W}(\mathcal{G})$ are the Weyl groups of the regular semisimple subalgebras of $\mathcal{G}$. For the algebras A_n, B_n, C_n and G_2 the Coxeter class is the only primitive conjugacy class [26]. Concretely, it is the class of the cyclic permutation $(n+1)$ for $\mathbf{W}(A_n)$ and that of the negative cycle $(\bar{n})$ for $\mathbf{W}(B_n) \simeq \mathbf{W}(C_n)$. For $\mathbf{W}(D_n)$ the situation is more interesting. The primitive conjugacy classes are those containing two negative cycles, $(\bar{n}_1, \bar{n}_2)$ for any $n_1 \geq n_2 \geq 1$, $n_1 + n_2 = n$, and the Coxeter class is that of $n_2 = 1$. The classification of the conjugacy classes in $\mathbf{W}(\mathcal{G})$ described in [26] is closely related to the classification of the regular semisimple subalgebras of $\mathcal{G}$ treated by Dynkin [34]. In fact, it has been shown[1] in [26] that each conjugacy class of $\mathbf{W}(\mathcal{G})$ can be (in general non-uniquely) represented by an element $w \in \mathbf{W}(\mathcal{G})$ of the product form

$$w = w_1 \cdot w_2 \cdots w_r, \tag{4.4}$$

where w_k belongs to a primitive conjugacy class in the Weyl group $\mathbf{W}(\mathcal{G}_k)$ of the simple factor $\mathcal{G}_k$ $(k = 1, \ldots, r)$ of a regular semisimple subalgebra of $\mathcal{G}$,

$$\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_r \subset \mathcal{G}. \tag{4.5}$$

The Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ on which w given in (4.4) acts is a direct sum

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \cdots + \mathcal{H}_r + \mathcal{H}', \tag{4.6}$$

where $\mathcal{H}_k$ is a Cartan subalgebra of $\mathcal{G}_k$ and w acts as the identity on the subalgebra $\mathcal{H}' \subset \mathcal{H}$ which is orthogonal to $\mathcal{H}_k$ for $k = 1, \ldots, r$ and satisfies $\operatorname{rank} \mathcal{G} = (\sum_k \operatorname{rank} \mathcal{G}_k) + \dim \mathcal{H}'$. For the construction of the corresponding Heisenberg subalgebra, one needs to lift w to a finite order inner automorphism $\hat{w}$ of $\mathcal{G}$. Clearly, the required lift can be taken to have the form

$$\hat{w} = \exp\left(2i\pi \mathrm{ad} X\right), \quad X = X_1 + X_2 + \cdots + X_r, \tag{4.7}$$

where $X_k \in \mathcal{G}_k$ defines an appropriate lift $\hat{w}_k$ of w_k to a finite order inner automorphism of $\mathcal{G}_k$,

$$\hat{w}_k = \exp\left(2i\pi \mathrm{ad} X_k\right), \quad X_k \in \mathcal{G}_k. \tag{4.8}$$

Below X_k will be given explicitly. We are interested in the graded Heisenberg subalgebra $\tilde{\mathcal{H}}_{\hat{w}} = \eta[\ell(\mathcal{H}, \hat{w})] \subset \ell(\mathcal{G})$ associated to $\hat{w}$. The twisted homogeneous Heisenberg subalgebra $\ell(\mathcal{H}, \hat{w}) \subset \ell(\mathcal{G}, \hat{w})$ in (2.4) obviously has the direct sum structure

$$\ell(\mathcal{H}, \hat{w}) = \ell(\mathcal{H}_1, \hat{w}_1) + \ell(\mathcal{H}_2, \hat{w}_2) + \cdots + \ell(\mathcal{H}_r, \hat{w}_r) + \ell(\mathcal{H}'). \tag{4.9}$$

Using $\hat{w}$ in (4.7), the "untwisting" η in (2.3) induces a corresponding direct sum structure

$$\tilde{\mathcal{H}}_{\hat{w}} = \tilde{\mathcal{H}}_{1,\hat{w}_1} + \tilde{\mathcal{H}}_{2,\hat{w}_2} + \cdots + \tilde{\mathcal{H}}_{r,\hat{w}_r} + \ell(\mathcal{H}'), \tag{4.10}$$

where $\tilde{\mathcal{H}}_{k,\hat{w}_k} \subset \ell(\mathcal{G}_k)$ is the Heisenberg subalgebra associated to the finite order inner automorphism $\hat{w}_k$ of $\mathcal{G}_k$, and $\ell(\mathcal{H}') = \mathcal{H}' \otimes \mathbf{C}[\lambda, \lambda^{-1}]$. This leads to the two-step strategy for constructing the non-equivalent graded Heisenberg subalgebras of the loop algebras $\ell(\mathcal{G})$: *i)* construct all of

[1] This is shown in [26] for any simple Lie algebra including the exceptional ones.

the Heisenberg subalgebras corresponding to the primitive conjugacy classes in the Weyl groups of the simple Lie algebras; *ii)* the general case is then obtained by running over the regular semisimple subalgebras of $\mathcal{G}$ and inserting the "primitive Heisenberg subalgebras" from the first step into the factors. Although the presentation of a Heisenberg subalgebra provided by this scheme is not unique in general, it is very convenient in practice. In particular, this scheme defines a correspondence between the Heisenberg subalgebras of $\ell(\mathcal{G})$ possessing a graded regular element and certain regular semisimple subalgebras of the Lie algebra $\mathcal{G}$. In the case when $\mathcal{G}$ is a classical Lie algebra, the correspondence is summarized in Table 4.

The notations used in Table 4 are as follows. A simple factor $\mathcal{G}_k$ appearing in the regular reductive subalgebra in the third column of the table represents the Coxeter class of $\mathbf{W}(\mathcal{G}_k)$ as well as the principal Heisenberg subalgebra of $\ell(\mathcal{G}_k)$. Concerning the primitive conjugacy classes in the D_n case, recall from Table 2 that in addition to the Coxeter class the other "extreme case" $(\bar{p},\bar{p})$ also admits a regular eigenvector for $n=2p$. The term $\bar{D}_{2p}$ in Table 4 represents the conjugacy class $(\bar{p},\bar{p})$ of $\mathbf{W}(D_{2p})$ and the respective non-principal primitive Heisenberg subalgebra of $\ell(D_{2p})$. The term $\mathcal{H}'_k$ denotes a Cartan piece of dimension k, and its presence means that the subspace $\ell(\mathcal{H}'_k)$ of the homogeneous Heisenberg subalgebra $\ell(\mathcal{H})\subset\ell(\mathcal{G})$ is contained in $\tilde{\mathcal{H}}_{\hat{w}}$. Since explicit realizations of the principal Heisenberg subalgebra of $\ell(\mathcal{G})$ are known for every simple Lie algebra, an explicit realization of any Heisenberg subalgebra appearing in Table 4 may be obtained if one constructs one for the primitive case $\bar{D}_{2p}$. This will be provided in the next subsection.

Algebra	Conjugacy class	Regular subalgebra	ord($\hat{w}$)
A_{ps-1}	$(p,\ldots,p)$	$A_{p-1}+\ldots+A_{p-1}+\mathcal{H}'_{s-1}$	p
$A_{p(s-1)}$	$(p,\ldots,p,1)$	$A_{p-1}+\ldots+A_{p-1}+\mathcal{H}'_{s-1}$	$\mathrm{gcd}(2,p)p$
D_{ps}	$(p,\ldots,p)$, p odd	$A_{p-1}+\ldots+A_{p-1}+\mathcal{H}'_{s}$	p
$D_{p(s-1)+1}$	$(p,\ldots,p,1)$, p odd	$A_{p-1}+\ldots+A_{p-1}+\mathcal{H}'_{s}$	p
D_{ps}	$(\bar{p},\ldots,\bar{p})$, s even	$\bar{D}_{2p}+\ldots+\bar{D}_{2p}$	$2p$
$D_{p(s-1)+1}$	$(\bar{p},\ldots,\bar{p},\bar{1})$, s even	$\bar{D}_{2p}+\ldots+\bar{D}_{2p}+D_{p+1}$	$2p$
$D_{p(s-1)+1}$	$(\bar{p},\ldots,\bar{p},1)$, s odd	$\bar{D}_{2p}+\ldots+\bar{D}_{2p}+\mathcal{H}'_1$	$2p$
B_{ps}	$(p,\ldots,p)$, p odd	$A_{p-1}+\ldots+A_{p-1}+\mathcal{H}'_{s}$	p
B_{ps}	$(\bar{p},\ldots,\bar{p})$, s even	$\bar{D}_{2p}+\ldots+\bar{D}_{2p}$	$2p$
B_{ps}	$(\bar{p},\ldots,\bar{p})$, s odd	$\bar{D}_{2p}+\ldots+\bar{D}_{2p}+B_p$	$2p$
C_{ps}	$(p,\ldots,p)$, p odd	$A_{p-1}+\ldots+A_{p-1}+\mathcal{H}'_{s}$	p
C_{ps}	$(\bar{p},\ldots,\bar{p})$	$C_p+\ldots+C_p$	$2p$

Table 4: Heisenberg subalgebras possessing a graded regular element.
Here s is the number of cycles in the partition, p is a positive integer and $A_0=\emptyset$.

4.2 A connection to sl_2 embeddings

For any simple Lie algebra $\mathcal{G}$, there exists a celebrated relationship [35] between the Coxeter class of $\mathbf{W}(\mathcal{G})$ and the conjugacy class of the principal sl_2 subalgebra of $\mathcal{G}$, whose essence is that the lift of a Coxeter element $w_c \in \mathbf{W}(\mathcal{G})$ may be chosen as

$$\hat{w}_c = \exp\left(2i\pi \frac{\mathrm{ad}I_0}{N_c}\right), \tag{4.11}$$

where N_c is the Coxeter number and I_0 is the semisimple element of a principal sl_2 subalgebra of $\mathcal{G}$. This means that there exists $I_\pm \in \mathcal{G}$ so that

$$[I_0, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_0, \tag{4.12}$$

and I_0 has the form $I_0 = \frac{1}{2}\sum_{\alpha>0} H_\alpha$, where the $H_\alpha \in \bar{\mathcal{H}}$ are the Cartan generators associated to a system of positive roots $\alpha > 0$ with respect to a Cartan subalgebra $\bar{\mathcal{H}} \subset \mathcal{G}$. The Cartan subalgebra $\bar{\mathcal{H}} \subset \mathcal{G}$ is said to be "in apposition" to the Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ on which w_c acts[2]. A consequence of this is that the grading of $\ell(\mathcal{G})$ induced by its isomorphism with $\ell(\mathcal{G}, \hat{w}_c)$ is the principal grading defined by

$$d_{N_c, I_0} = N_c \lambda \frac{d}{d\lambda} + \mathrm{ad}I_0. \tag{4.13}$$

Furthermore, decomposing $\mathcal{G}$ as

$$\mathcal{G} = \mathcal{G}^{I_0}_{<0} + \mathcal{G}^{I_0}_{0} + \mathcal{G}^{I_0}_{>0} \tag{4.14}$$

using the (principal) grading of $\mathcal{G}$ defined by $\mathrm{ad}I_0$, the grade 1 regular element Λ of the principal Heisenberg subalgebra $\tilde{\mathcal{H}}_{\hat{w}_c}$ takes the form

$$\Lambda = C_+ + \lambda C_-, \quad C_\pm \in \mathcal{G}, \quad \text{with} \quad C_+ = I_+, \tag{4.15}$$

i.e., the sl_2 subalgebra of $\mathcal{G}$ defined by the nilpotent element $C_+ \in \mathcal{G}$ through the Jacobson-Morozov theorem [42] is the same sl_2 that enters the grading (4.13). Note also that

$$[C_-, \mathcal{G}^{I_0}_{<0}] = \{0\}. \tag{4.16}$$

The relations expressed by formulas (4.11), (4.15), (4.16) play an important role in the Drinfeld-Sokolov construction of KdV type hierarchies and we wish to show that they generalize to all cases given in Table 4, for which a graded regular element exists in the Heisenberg subalgebra. (The case of the homogeneous Heisenberg subalgebra is related to the trivial, identically zero, sl_2 embedding and is excluded in what follows.) We need to deal with the $\bar{D}_{2p}$ case first, since it occurs as a "building block" in Table 4.

In order to take care of the $\bar{D}_{2p}$ case, we make use of a result of [39] on the lift of a Weyl group element $w_{(\bar{n}_1,\bar{n}_2)} \in \mathbf{W}(D_n)$ belonging to the conjugacy class $(\bar{n}_1, \bar{n}_2)$. In Section 2.6 of [39], a lift $\hat{w}_{(n_1,n_2)}$ conjugate to

$$\hat{\tau}_{(\bar{n}_1,\bar{n}_2)} := \exp\left(2i\pi \mathrm{ad}K/N\right), \tag{4.17}$$

[2] Equivalently, if the principal sl_2 generator I_0 is taken from $\mathcal{H}$ then $\hat{w}_c$ defined by (4.11) acts as a Coxeter element on the Cartan subalgebra in apposition $\bar{\mathcal{H}}$, which may be defined as the centralizer of an element $(I_+ + C_-) \in \mathcal{G}$, where $C_- \neq 0$ is chosen in such a way that $[I_0, C_-] = -(N_c - 1)C_-$.

where $N = \mathrm{lcm}(2n_1, 2n_2)$ is the order of $\hat{\tau}_{(\bar{n}_1,\bar{n}_2)}$ and

$$K = \frac{N}{2n_1}\sum_{k=1}^{n_1}(n_1 - k + 1)\epsilon_k + \frac{N}{2n_2}\sum_{k=1}^{n_2}(n_2 - k)\epsilon_{n_1+k}, \tag{4.18}$$

was constructed for any $n_1 + n_2 = n$. We observe that K is the semisimple element of an sl_2 subalgebra of D_n in the Coxeter case $n_2 = 1$ and in the case $n_1 = n_2$, and is not proportional to such an element in the other cases. This is most easily seen from the spectrum of the matrix K in the defining $2n$ dimensional representation of D_n, taking into account that ϵ_k $(k = 1, \ldots, n)$ contains two non-zero entries, ± 1, when diagonalized. For $n_1 = n_2 = p$, this explicit form of K also implies that the $4p$ dimensional vector representation of D_{2p} decomposes under the sl_2 subalgebra containing K according to

$$4p = (2p + 1) + (2p - 1). \tag{4.19}$$

According to Dynkin [34], this is one of the singular sl_2 subalgebras ("S-subalgebras") in D_{2p}. (Note that the singular sl_2 subalgebras of [34] are called semi-regular sl_2 subalgebras and the principal sl_2 is called the regular sl_2 in some of the literature.) It is interesting that the number of conjugacy classes of singular sl_2 subalgebras in D_n is actually equal to the number of primitive conjugacy classes in $\mathbf{W}(D_n)$, but the above lift of $w_{(\bar{n}_1,\bar{n}_2)}$ corresponds to an sl_2 embedding only in the cases when $w_{(\bar{n}_1,\bar{n}_2)}$ admits a regular eigenvector.

It follows from the above that $\hat{\tau}_{(\bar{p},\bar{p})}$ in (4.17) may be used in the construction of the sought after Heisenberg subalgebra, denoted as $\tilde{\mathcal{H}}_{(\bar{p},\bar{p})}$, where K is the semisimple sl_2 generator corresponding to the decomposition (4.19) of the defining representation of D_{2p} (which defines it up to conjugation). It is convenient to realize D_{2p} as the subalgebra of gl_{4p} consisting of the matrices A subject to $A^t\eta + \eta A = 0$ with the $4p \times 4p$ matrix η given by

$$\eta := \sum_{k=1}^{2p+1} e_{k,2p+2-k} + \sum_{k=1}^{2p-1} e_{2p+1+k,4p+1-k}, \tag{4.20}$$

where $e_{i,j}$ is the usual elementary matrix with a single non-zero entry 1 at the ij position, and to realize the sl_2 generator K as

$$K = \mathrm{diag}\,(p, \ldots, 0, \ldots, -p, (p-1), \ldots, 0, \ldots, -(p-1))\,. \tag{4.21}$$

The appropriate grading of $\ell(D_{2p})$ is given by

$$d_{2p,K} = 2p\lambda\frac{d}{d\lambda} + \mathrm{ad}K. \tag{4.22}$$

Note also from Table 2 that the grade q subspace of $\tilde{\mathcal{H}}_{(\bar{p},\bar{p})}$ must be of dimension 2 if $q = 1, 3, \ldots, (2p-1)$ modulo $2p$, and is otherwise empty. Let us now introduce the matrices $H_{1,1}$ and $H_{1,2}$ in D_{2p},

$$H_{1,1} := \sum_{k=1}^{p} a_k e_{k,k+1} - \sum_{k=1}^{p} a_{p+1-k} e_{p+k,p+k+1} + a_{p+1}(e_{2p,1} - e_{2p+1,2}), \tag{4.23}$$

$$\begin{aligned} H_{1,2} := &\sum_{k=1}^{p-1} b_k e_{2p+k+1,2p+k+2} - \sum_{k=1}^{p-1} b_{p-k} e_{3p+k,3p+k+1} + b_p a_1(e_{4p,2p+1} - e_{1,2p+2}) + \\ &b_p a_{p+1}(e_{4p,1} - e_{2p+1,2p+2}), \end{aligned}$$

where $a_1, \ldots, a_{p+1}, b_1, \ldots, b_p \in \mathbf{C}$ are arbitrarily chosen *non-zero* parameters. We also need their matrix powers

$$H_{j,k} := (H_{1,k})^{2j-1} \quad \text{for} \quad j = 1, 2, \ldots, p, \quad k = 1, 2. \tag{4.24}$$

It can be checked that these $2p$ matrices commute and span a Cartan subalgebra of D_{2p} for generic choice of the parameters. We denote this Cartan subalgebra as $\mathcal{H}_{(\bar{p},\bar{p})}$. The point is that $\mathcal{H}_{(\bar{p},\bar{p})} \subset D_{2p}$ is invariant under the automorphism given in (4.17), and $H_{j,k}$ is the corresponding basis of eigenvectors:

$$\hat{\tau}_{(\bar{p},\bar{p})}\left(H_{j,k}\right) = (\omega)^{2j-1} H_{j,k}, \quad \omega = \exp\left(\frac{2i\pi}{N}\right), \quad N = 2p. \tag{4.25}$$

This implies that $\hat{\tau}_{(\bar{p},\bar{p})}$ acts on the Cartan subalgebra $\mathcal{H}_{(\bar{p},\bar{p})}$ as a representative of the conjugacy class $(\bar{p},\bar{p}) \subset \mathbf{W}(D_{2p})$. Performing the "untwisting" described in Section 2 is straightforward, and we get the Heisenberg subalgebra $\tilde{\mathcal{H}}_{(\bar{p},\bar{p})} \subset \ell(D_{2p})$ as the span of the following graded basis:

$$\lambda^m \left(\Lambda_{1,k}\right)^{2j-1}, \quad \text{for} \quad m \in \mathbf{Z},\ j = 1, 2, \ldots, p,\ k = 1, 2, \tag{4.26}$$

where

$$\begin{aligned}
\Lambda_{1,1} &:= \sum_{k=1}^{p} a_k e_{k,k+1} - \sum_{k=1}^{p} a_{p+1-k} e_{p+k,p+k+1} + \lambda a_{p+1}(e_{2p,1} - e_{2p+1,2}), \\
\Lambda_{1,2} &:= \sum_{k=1}^{p-1} b_k e_{2p+k+1,2p+k+2} - \sum_{k=1}^{p-1} b_{p-k} e_{3p+k,3p+k+1} + b_p a_1(e_{4p,2p+1} - e_{1,2p+2}) + \\
&\quad \lambda b_p a_{p+1}(e_{4p,1} - e_{2p+1,2p+2}).
\end{aligned} \tag{4.27}$$

The basis vector in (4.26) has grade $(2j-1) + 2mp$ with respect to the grading $d_{2p,K}$. This construction of $\tilde{\mathcal{H}}_{(\bar{p},\bar{p})}$ was inspired by an analogous construction in [39]. A grade 1 regular element $\Lambda \in \tilde{\mathcal{H}}_{(\bar{p},\bar{p})}$ will be a linear combination

$$\Lambda = d_1 \Lambda_{1,1} + d_2 \Lambda_{1,2} \tag{4.28}$$

with generic non-zero coefficients d_1, d_2. Writing Λ in the form $\Lambda = C_+ + \lambda C_-$, C_+ has grade 1 and C_- has grade $-(2p-1)$ with respect to ad$K$. We wish to show that K and C_+ are contained in the same sl_2 subalgebra of D_{2p}, i.e., that the commutation relations given in (4.12) hold with $I_0 := K$, $I_+ := C_+$ and some $I_- \in D_{2p}$, analogously to the principal case.

We need to present an auxiliary result at this point. Consider a regular semisimple element $\Lambda = (C_+ + \lambda C_-) \in \ell(\mathcal{G})$, with some $C_\pm \in \mathcal{G}$, having definite grade with respect to a grading operator $d_{N,K} = N\lambda \frac{d}{d\lambda} + \mathrm{ad}K$. Suppose that

$$[C_-, \mathcal{G}^K_{<0}] = \{0\}, \tag{4.29}$$

where $\mathcal{G} = \mathcal{G}^K_{>0} + \mathcal{G}^K_0 + \mathcal{G}^K_{<0}$ is the decomposition defined by means of the eigenvalues of ad$K$. Then the following "non-degeneracy relation"

$$\mathrm{Ker}\Big(\mathrm{ad}\, C_+\Big) \cap \mathcal{G}^K_{<0} = \{0\} \tag{4.30}$$

is satisfied. Indeed, if one could find an element $v \in \mathcal{G}^K_{<0}$ for which $[C_+, v] = 0$, then $[\Lambda, v] = 0$ would also hold because of (4.29). Clearly, $\mathrm{Ker}\,(\mathrm{ad}\Lambda) \subset \ell(\mathcal{G})$ can contain only semisimple elements of $\mathcal{G} \subset \ell(\mathcal{G})$, but any $v \in \mathcal{G}^K_{<0}$ is a nilpotent element. This contradiction proves (4.30).

The above argument applies to Λ in (4.28), since (4.29) follows from the fact that the grade of C_- is the smallest eigenvalue of $\mathrm{ad}K$ on $\mathcal{G} = D_{2p}$. A consequence of the non-degeneracy relation (4.30) is the equality $\dim\,[C_+, \mathcal{G}^K_{-1}] = \dim \mathcal{G}^K_{-1}$. This implies the existence of $I_- \in \mathcal{G}^K_{-1}$ for which $[C_+, I_-] = K$, since in our case $\dim \mathcal{G}^K_{-1} = \dim \mathcal{G}^K_0$ holds as is easily verified using the explicit formula (4.21) of the grading operator K. The set $\{I_-, I_0 := K, I_+ := C_+\}$ spans the required sl_2 subalgebra. This settles the $\bar{D}_{2p}$ case.

Turning now to the general case, we first rewrite the lift $\hat{w}$ in (4.7) as

$$\hat{w} = \exp\left(2i\pi \frac{\mathrm{ad}Y}{N}\right), \tag{4.31}$$

where

$$Y = NX = \frac{N}{N_1}Y_1 + \frac{N}{N_2}Y_2 + \cdots + \frac{N}{N_r}Y_r. \tag{4.32}$$

Here N is the order of $\hat{w}$, N_k is the order of $\hat{w}_k$ when acting on $\mathcal{G}_k$, $Y_k = N_k X_k$ in (4.8). The grading of $\ell(\mathcal{G})$ corresponding to $\hat{w}$ is defined by the operator $d_{N,Y}$,

$$d_{N,Y} = N\lambda \frac{d}{d\lambda} + \mathrm{ad}\,Y. \tag{4.33}$$

When restricted to the subalgebra $\ell(\mathcal{G}_k)$, this grading satisfies

$$d_{N,Y}|_{\ell(\mathcal{G}_k)} = \frac{N}{N_k} d_{N_k,Y_k}, \quad d_{N_k,Y_k} = N_k \lambda \frac{d}{d\lambda} + \mathrm{ad}\,Y_k, \tag{4.34}$$

where d_{N_k,Y_k} gives the grading of $\ell(\mathcal{G}_k)$ induced by the isomorphism $\ell(\mathcal{G}_k) \simeq \ell(\mathcal{G}_k, \hat{w}_k)$. Using the lifts of the regular primitive Weyl transformations given in (4.11) and in (4.17), Y_k is the semisimple element of an sl_2 subalgebra of $\mathcal{G}_k$ with the same normalization as I_0 in (4.12). Hence it follows from (4.32) that Y is proportional to the semisimple element of an sl_2 subalgebra of $\mathcal{G}$ if and only if

$$N_i = N_j \quad \text{for any} \quad i \neq j. \tag{4.35}$$

Inspection shows that (4.35) is satisfied for all cases in Table 4, and therefore

$$Y = \frac{N}{N_1}(Y_1 + Y_2 + \cdots + Y_r), \tag{4.36}$$

where $\frac{N}{N_1}$ turns out to be 1 or 2 depending on whether $(Y_1 + Y_2 + \cdots + Y_r)$ defines an integral or a half-integral sl_2 subalgebra of $\mathcal{G}$, i.e., whether the grading of $\mathcal{G}$ defined by this element is integral or half-integral. In fact, the sl_2 embedding is an integral one in all cases in Table 4 except the case $(p, \ldots, p, 1)$ with p *even* for $\mathcal{G} = A_{p(s-1)}$. One also sees that any graded regular semisimple element $\Lambda \in \tilde{\mathcal{H}}_{\hat{w}}$ of minimal positive grade (N/N_1) has the form $\Lambda = C_+ + \lambda C_-$, where $I_+ := C_+$ is contained in an sl_2 subalgebra whose semisimple element is $I_0 := \frac{N_1}{N}Y$ given by (4.32). This is a consequence of what we know about the principal and $\bar{D}_{2p}$ cases, simply because such a Λ is a linear combination of respective graded regular elements from the

Heisenberg subalgebras $\tilde{\mathcal{H}}_{\hat{w}_k} \subset \ell(\mathcal{G}_k)$ in (4.10). With respect to the grading of $\mathcal{G}$ defined by $\mathrm{ad}I_0$, the non-degeneracy relation

$$\mathrm{Ker}\Big(\mathrm{ad}\, I_+\Big) \cap \mathcal{G}^{I_0}_{<0} = \{0\} \tag{4.37}$$

then follows from the sl_2 structure. Inspection shows that $[C_-, \mathcal{G}^{I_0}_{<0}] = \{0\}$ is also satisfied in each case, since C_- is an eigenvector of $\mathrm{ad}I_0$ associated to the smallest eigenvalue.

Let us summarize the results obtained in this section. For $\mathcal{G}$ a classical Lie algebra, we verified the following connection between regular conjugacy classes in $\mathbf{W}(\mathcal{G})$, with graded regular elements in the associated Heisenberg subalgebra $\tilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$, and sl_2 subalgebras in $\mathcal{G}$. For any regular conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$, the appropriately lifted Weyl transformation takes the form $\hat{w} = \exp\left(2i\pi \mathrm{ad}Y/N\right)$ in (4.31), where $Y = \nu I_0$ with I_0 being the semisimple generator of an sl_2 subalgebra of $\mathcal{G}$ and $\nu = 1$ or $\nu = 2$ so that $\mathrm{ad}Y$ has integral eigenvalues. The largest eigenvalue of $\mathrm{ad}Y$ is $(N-\nu)$, where N is the order of $\hat{w}$ and $m = N/\nu$ is the order of $w \in [w]$. The smallest positive d_{m,I_0} grade for which a graded regular element $\Lambda \in \tilde{\mathcal{H}}_{\hat{w}}$ exists is one, and any grade one regular element has the form $\Lambda = (C_+ + \lambda C_-)$, where C_+ is included in an sl_2 subalgebra containing also I_0. The eigenvalue of $\mathrm{ad}I_0$ is minimal on C_-. Of course $\hat{w}$ acts as the Weyl transformation w on the Cartan subalgebra defined by the centralizer of its regular semisimple eigenvector given by $H := \Lambda(\lambda = 1) = (C_+ + C_-) \in \mathcal{G}$.

If w in (4.4) is a Coxeter element in a regular semisimple subalgebra of $\mathcal{G}$, the above results followed from the result of Kostant [35] on the connection between the Coxeter class and the principal sl_2 given by formula (4.11). The case of the regular semi-Coxeter conjugacy class $(\bar{p}, \bar{p}) \subset \mathbf{W}(D_{2p})$ was dealt with by inspecting the lift found in [39].

We wish to note that in [27] the result of Kostant [35] was generalized to give a similar connection between certain regular conjugacy classes in $\mathbf{W}(\mathcal{G})$ and those special integral sl_2 subalgebras of $\mathcal{G}$ for which the decomposition of $\mathcal{G}$ into sl_2 irreducible representations contains no singlets and only one triplet. In addition to the principal sl_2, such sl_2 subalgebras exist only in the exceptional Lie algebras as listed in [27] [3]. See also Appendix A.

In passing, we also wish to mention the correspondence found in [40] between the conjugacy classes of arbitrary singular (semi-regular) sl_2 subalgebras in $\mathcal{G}$ [34] and a subset of the primitive (semi-Coxeter) conjugacy classes in $\mathbf{W}(\mathcal{G})$. This is given in terms of an injective mapping from the set of singular sl_2 subalgebras into the set of primitive conjugacy classes, which is defined by the coincidence of the so called "Carter diagrams" [26] associated to the conjugacy classes in $\mathbf{W}(\mathcal{G})$ and to the sl_2 subalgebras in $\mathcal{G}$. On the overlap of their ranges of applicability, the "Kostant type" correspondence discussed in [27], and here for D_{2p}, and the one in [40] are consistent. It is not clear whether the result of [40] has any significance for the theory of integrable hierarchies.

[3] The sl_2 subalgebra of G_2 appearing in Table 11 of [27] has in fact 3 triplets and not 1, but the claims are still valid for this sl_2 as is easily seen using that it is actually the principal sl_2 inside the regular $A_2 \subset G_2$.

5 Applications to KdV type systems

Now we turn to the application of the results collected in the previous sections to the construction of integrable hierarchies. For $\mathcal{G}$ a simple Lie algebra, fix a grade one regular semisimple element Λ from a Heisenberg subalgebra $\widetilde{\mathcal{H}}_{\hat{w}} \subset \ell(\mathcal{G})$. Suppose that Λ has the form

$$\Lambda = I_+ + \lambda C_-, \tag{5.1}$$

where I_+ belongs to the sl_2 subalgebra $\{I_-, I_0, I_+\} \subset \mathcal{G}$ for which d_{m,I_0} in (4.1) defines the grading of $\ell(\mathcal{G})$. Suppose also that

$$[C_-, \mathcal{G}_{<0}] = \{0\} \qquad \text{with} \qquad \mathcal{G}_{<0} = \sum_{k<0} \mathcal{G}_k, \tag{5.2}$$

where $\mathcal{G}_k$, denoted in Section 4 as $\mathcal{G}_k^{I_0}$, is the eigensubspace of $\mathrm{ad}I_0$ with eigenvalue k.

As we have seen, for $\mathcal{G}$ a classical Lie algebra the relations in (5.1) and (5.2) are ensured by using the lift $\hat{w}$ given in (4.31) for an arbitrary regular conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$. For the exceptional Lie algebras these relations may be assumed in connection with many regular conjugacy classes in the Weyl group, which include for example all regular conjugacy classes in $\mathbf{W}(G_2)$ and all of the regular primitive conjugacy classes. It appears that in $\mathbf{W}(F_4)$, $\mathbf{W}(E_{6,7,8})$ there exist some regular conjugacy classes for which (5.2) cannot be satisfied; see Appendix A.

Let us recall [30, 31] that one may associate a "classical $\mathcal{W}$-algebra" to any sl_2 subalgebra of $\mathcal{G}$ by a generalization of the Hamiltonian reduction used by Drinfeld and Sokolov to obtain the second (Gelfand-Dickey) Poisson bracket of their KdV type hierarchies. In Subsection 5.1 we show that if the sl_2 subalgebra is related to a grade one regular semisimple element Λ in the above way, which specifies a (small) subset of the non-equivalent sl_2 subalgebras of $\mathcal{G}$, then it is possible to obtain a KdV type hierarchy from Hamiltonian reduction whose second Poisson bracket is the $\mathcal{W}$-algebra defined by the sl_2-subalgebra. Subsection 5.2 is devoted to the concrete description of some of the systems that may be obtained from this approach. We analyze the cases when $\mathcal{G}$ is a classical Lie algebra of B, C or D type and the regular reductive subalgebra appearing in the third column of Table 4 contains only A or C type simple factors. The resulting generalized KdV systems turn out to be discrete reductions of the systems associated to gl_n having the Gelfand-Dickey type Lax operators in (1.3) and (1.4). That is the Lax operators of the resulting systems are of the form (1.3) or (1.4) subject to certain extra symmetry conditions, very much like the well-known principal case [1] for the Lie algebra C_p, where the Lax operator is of the form (1.1) with $n = 2p$ subject to the self-adjointness condition $L^\dagger = L$.

5.1 KdV systems associated to grade one regular elements

The following construction is a straightforward generalization of that in [1], and can be also viewed as a special case of the more general construction given in [11, 12, 13].

After fixing a grade one regular semisimple element $\Lambda \in \ell(\mathcal{G})$ subject to (5.1), (5.2), consider the manifold $\mathcal{M}$ consisting of first order differential operators,

$$\mathcal{M} := \left\{ \mathcal{L} = \partial + J + \lambda C_- \mid J \in C^\infty\left(S^1, \mathcal{G}\right) \right\}. \tag{5.3}$$

The manifold $\mathcal{M}$ is the phase space of an infinite collection of bi-Hamiltonian systems. The two compatible Poisson brackets (PBs) are given as follows. The "second" PB is given by the affine current algebra structure,

$$\{f, h\}_2(J) = \int_{S^1} \mathrm{tr}\left(J \left[\frac{\delta f}{\delta J}, \frac{\delta h}{\delta J} \right] + \left(\frac{\delta f}{\delta J} \right)' \frac{\delta h}{\delta J} \right), \tag{5.4}$$

and the "first" PB is given by

$$\{f, h\}_1(J) = -\int_{S^1} \mathrm{tr}\, C_- \left[\frac{\delta f}{\delta J}, \frac{\delta h}{\delta J} \right], \tag{5.5}$$

for f, h smooth functions on $\mathcal{M}$. The Hamiltonians of interest are generated by the invariants ("eigenvalues") of the monodromy matrix $T(J, \lambda)$ of $\mathcal{L}$. The corresponding Hamiltonian flows commute as a special case of the Adler-Kostant-Symes construction and are bi-Hamiltonian (see e.g. [43]) . The Hamiltonians given by the monodromy invariants are non-local functionals of J in general. Using that C_- in (5.3) is related to the regular semisimple element Λ according to (5.1), we can perform a symmetry reduction of the system on $\mathcal{M}$ leading to a *local* hierarchy.

Let G be a connected Lie group corresponding to $\mathcal{G}$. Define the subgroup $\mathrm{Stab}(C_-)$ of G by $gC_-g^{-1} = C_-$ for $g \in \mathrm{Stab}(C_-)$. Denote the group of smooth loops based on $\mathrm{Stab}(C_-)$ as $\widetilde{\mathrm{Stab}}(C_-) := C^\infty(S^1, \mathrm{Stab}(C_-))$. The possibility for reduction rests upon the fact that there is a Poisson action (meaning that it leaves the PBs unchanged) of $\widetilde{\mathrm{Stab}}(C_-)$ on $\mathcal{M}$ given by

$$(\partial + J + \lambda C_-) \mapsto g(\partial + J + \lambda C_-)g^{-1} = g(\partial + J)g^{-1} + \lambda C_-, \quad \forall\, g \in \widetilde{\mathrm{Stab}}(C_-), \tag{5.6}$$

which leaves the monodromy invariants unchanged. For present purposes we consider reduction based on the subgroup $\mathcal{N}$ of $\widetilde{\mathrm{Stab}}(C_-)$ whose Lie algebra is $C^\infty(S^1, \mathcal{G}_{<0})$. The reduction is defined by first imposing constraints on $\mathcal{M}$ so that the constrained submanifold $\mathcal{M}_c \subset \mathcal{M}$ is

$$\mathcal{M}_c := \left\{ \mathcal{L} = \partial + j + \Lambda \,|\, j \in C^\infty\left(S^1, \mathcal{G}_{<1}\right) \right\}, \qquad \left(\mathcal{G}_{<1} = \sum_{k<1} \mathcal{G}_k \right). \tag{5.7}$$

That is the constraints defining $\mathcal{M}_c \subset \mathcal{M}$ restrict J to have the form $J = (j + I_+)$ with I_+ in (5.1). The second step of the reduction is to factorize $\mathcal{M}_c$ by the group $\mathcal{N}$ of "gauge transformations" acting according to

$$e^f : \mathcal{L} \mapsto e^f \mathcal{L} e^{-f}, \qquad \forall\, e^f \in \mathcal{N}, \quad \text{with} \quad f \in C^\infty\left(S^1, \mathcal{G}_{<0}\right). \tag{5.8}$$

Standard arguments show that the compatible PBs on $\mathcal{M}$ induce compatible PBs on the space of gauge invariant functions on $\mathcal{M}_c$, identified as the space of functions on the reduced space $\mathcal{M}_{\mathrm{red}} = \mathcal{M}_c/\mathcal{N}$. Thanks to the non-degeneracy relation in (4.37), the gauge fixing procedure of [1] is applicable to obtain a basis of the gauge invariant differential polynomials on $\mathcal{M}_c$, which may be used as coordinate functions on $\mathcal{M}_c/\mathcal{N}$. The gauges resulting from this procedure are often called "DS gauges" (see e.g. [31]). A particular DS gauge is the so called lowest weight gauge [44], whose gauge section $\mathcal{M}_{\mathrm{l.w.}} \subset \mathcal{M}_c$ is defined as

$$\mathcal{M}_{\mathrm{l.w.}} := \left\{ \mathcal{L} = \partial + j_{\mathrm{l.w.}} + \Lambda \,|\, j_{\mathrm{l.w.}} \in C^\infty\left(S^1, \mathrm{Ker}\,(\mathrm{ad}\, I_-)\right) \right\}. \tag{5.9}$$

In terms of the one-to-one model of $\mathcal{M}_c/\mathcal{N}$ furnished by the global gauge section $\mathcal{M}_{\rm l.w.}$, the reduced second PB is given by the Dirac bracket algebra of the components of $j_{\rm l.w.}$ induced from (5.4). This Dirac bracket algebra is just the classical $\mathcal{W}$-algebra of [30] (see also [31]) associated to the sl_2 subalgebra $\{I_-, I_0, I_+\} \subset \mathcal{G}$.

A generalized KdV hierarchy of bi-Hamiltonian flows is generated on the reduced space $\mathcal{M}_c/\mathcal{N}$ by the commuting Hamiltonians provided by the local monodromy invariants of $\mathcal{L}$, which are determined through the abelianization procedure described in equations (1.9), (1.10).

The hierarchy on $\mathcal{M}_{\rm red}$ resulting from the above "DS type" symmetry reduction [1] often possesses a residual symmetry that may be used to reduce it further. Define the subgroup G_R of $\mathrm{Stab}(C_-)$ by

$$G_R := \mathrm{Stab}\,(C_-) \cap \mathrm{Stab}\,(I_+) \cap \mathrm{Stab}\,(I_-)\,. \tag{5.10}$$

Let $\{T_a\}$ denote a basis of the Lie algebra $\mathcal{G}_R$ of G_R,

$$\mathcal{G}_R = \mathrm{Ker}\,(\mathrm{ad}\,C_-) \cap \mathrm{Ker}\,(\mathrm{ad}\,I_+) \cap \mathrm{Ker}\,(\mathrm{ad}\,I_-)\,. \tag{5.11}$$

In fact the subgroup $\widetilde{G_R} := C^\infty\,(S^1, G_R)$ of $\widetilde{\mathrm{Stab}(C_-)}$ survives the DS type symmetry reduction. Taking $\mathcal{M}_{\rm l.w.}$ as the model of $\mathcal{M}_c/\mathcal{N}$, the residual $\widetilde{G_R}$ symmetry acts as

$$(\partial + j_{\rm l.w.} + \Lambda) \mapsto g(\partial + j_{\rm l.w.} + \Lambda)g^{-1} = g(\partial + j_{\rm l.w.})g^{-1} + \Lambda, \quad \forall\, g \in \widetilde{G_R}. \tag{5.12}$$

These transformations leave invariant the compatible PBs and the commuting Hamiltonians constituting the KdV type system on $\mathcal{M}_{\rm l.w.}$. At the infinitesimal level, the $\widetilde{G_R}$ symmetry in (5.12) is generated through the second ($\mathcal{W}$-algebra) PB by the components $\mathrm{tr}\,(T_a j_{\rm l.w.})$ of $j_{\rm l.w.}$, that is by the subset of the sl_2 singlet components of $j_{\rm l.w.}$ annihilated by $\mathrm{ad}\,C_-$.

The residual symmetry in (5.12) is a continuous symmetry. Another interesting possibility, which is important in examples as we shall see later, is the presence of a discrete symmetry. This occurs for instance in the following situation. Let $\gamma : \mathcal{G} \to \mathcal{G}$ be an involutive automorphism with a corresponding involution $\Gamma : G \to G$. In the obvious way, extend γ to an involution of $\ell(\mathcal{G})$. Suppose now that Λ is a grade one regular semisimple element of $\ell(\mathcal{G})$ which is γ-invariant, $\gamma(\Lambda) = \Lambda$, and the grading d_{m,I_0} is also invariant, $\gamma(I_0) = I_0$. Suppose furthermore that the fixed point set $\mathcal{G}^\gamma \subset \mathcal{G}$ is a simple Lie algebra. (All classical Lie algebras are fixed point sets in gl_n, or sl_n, for appropriate γ.) The Heisenberg subalgebra $\tilde{\mathcal{H}}_\Lambda := \mathrm{Ker}\,(\mathrm{ad}\,\Lambda)$ of $\ell(\mathcal{G})$ is an invariant subspace of γ, and the fixed point set $\tilde{\mathcal{H}}_\Lambda^\gamma \subset \tilde{\mathcal{H}}_\Lambda$ is a Heisenberg subalgebra of $\ell(\mathcal{G}^\gamma)$. We can now perform the above DS reduction leading to a KdV type hierarchy using the same element Λ and either a system based on $\mathcal{G}$ or one based on $\mathcal{G}^\gamma$ as the original system.

In the former case we start with the bi-Hamiltonian manifold $\mathcal{M}$ in (5.3), introduce the constrained manifold $\mathcal{M}_c$ in (5.7), and end up with $\mathcal{M}_{\rm red} = \mathcal{M}_c/\mathcal{N}$. The natural action of γ on $\mathcal{M}$, given by

$$\gamma : (\partial + J + \lambda C_-) \mapsto (\partial + \gamma(J) + \lambda C_-), \tag{5.13}$$

leaves invariant the compatible PBs on $\mathcal{M}$. Since $\mathcal{M}_c$ is mapped to itself by γ and also $\mathcal{N}$ is mapped to itself by Γ as γ preserves the grading, the action (5.13) induces a corresponding action of γ on $\mathcal{M}_{\rm red}$. On account of $\gamma(I_-) = I_-$, which may be assumed by choosing I_-, the gauge section $\mathcal{M}_{\rm l.w.}$ of the $\mathcal{N}$ orbits in $\mathcal{M}_c$, defined in (5.9), is mapped to itself by γ in (5.13). Hence in terms of the model $\mathcal{M}_{\rm l.w.}$ of $\mathcal{M}_{\rm red}$ the induced action is simply given by

$$\gamma : (\partial + j_{\rm l.w.} + \Lambda) \mapsto (\partial + \gamma(j_{\rm l.w.}) + \Lambda). \tag{5.14}$$

The action on $\mathcal{M}_{\rm red} = \mathcal{M}_c/\mathcal{N} \simeq \mathcal{M}_{\rm l.w.}$ given by (5.14) leaves invariant the compatible PBs induced from those in (5.4), (5.5) by means of the DS reduction. Recall that the Hamiltonian densities yielding the commuting Hamiltonians of the KdV type hierarchy on $\mathcal{M}_{\rm red}$ are the components of $h(j) \in \tilde{\mathcal{H}}_\Lambda$ defining the "abelianized" form $(\partial + h(j) + \Lambda)$ of $\mathcal{L} = (\partial + j + \Lambda) \in \mathcal{M}_c$. The uniqueness property of the abelianization procedure in (1.9), (1.10) implies the equality

$$h\left(\gamma(j)\right) = \gamma\left(h(j)\right), \tag{5.15}$$

which means that the Hamiltonians corresponding to the components of $h(j)$ in $\tilde{\mathcal{H}}_\Lambda^\gamma$ are invariant, and those corresponding to the eigenvalue -1 of γ on the Heisenberg subalgebra $\tilde{\mathcal{H}}_\Lambda$ are "anti-invariant" (change sign) under the action of γ. Since the PBs are γ-invariant, the Hamiltonian flows on $\mathcal{M}_{\rm red}$ generated by the γ-invariant Hamiltonians preserve the fixed point set $\mathcal{M}_{\rm red}^\gamma \subset \mathcal{M}_{\rm red}$ of γ. (The "anti-invariant" Hamiltonians vanish on the fixed point set and the Hamiltonian flows defined by them are transversal to it.) Therefore we can define a hierarchy on $\mathcal{M}_{\rm red}^\gamma$ by restricting the flows of the hierarchy generated on $\mathcal{M}_{\rm red}$ by the γ-invariant Hamiltonians to $\mathcal{M}_{\rm red}^\gamma$. The flows of the resulting hierarchy are Hamiltonian with respect to the compatible PBs on the space $\mathcal{M}_{\rm red}^\gamma \simeq \mathcal{M}_{\rm l.w.}^\gamma$ obtained from those on $\mathcal{M}_{\rm l.w.}$ by restricting the PBs of the γ-invariant components of $j_{\rm l.w.}$, which may be regarded as coordinates on $\mathcal{M}_{\rm l.w.}^\gamma$, to this fixed point set. We refer to the reduction procedure just given as "discrete reduction".

Using the gauge group $\mathcal{N}^{\rm T}$ whose Lie algebra is $C^\infty\left(S^1, \mathcal{G}_{<0}^\gamma\right)$, we can also perform the above discussed DS type reduction of the system on $\mathcal{M}^\gamma$,

$$\mathcal{M}^\gamma = \Big\{ \mathcal{L} = \partial + J + \lambda C_- \mid J \in C^\infty\Big(S^1, \mathcal{G}^\gamma\Big) \Big\}. \tag{5.16}$$

The system on $\mathcal{M}^\gamma$ consists of the compatible Poisson brackets, defined similarly to (5.4) and (5.5) using $\mathcal{G}^\gamma$ in place of $\mathcal{G}$, and the monodromy invariants. Here the invariant scalar product "tr" on $\mathcal{G}^\gamma \subset \mathcal{G}$ is taken to be the restriction of that on $\mathcal{G}$. Clearly, the system on $\mathcal{M}^\gamma$ may be obtained by discrete reduction from the system on $\mathcal{M}$. The discrete reduction of $\mathcal{M}$ to $\mathcal{M}^\gamma$ induces the discrete reduction of $\mathcal{M}_{\rm red}$ to $\mathcal{M}_{\rm red}^\gamma$. We then have the following result.

Proposition 5.1. *The hierarchy on $\mathcal{M}_{\rm red}^\gamma$ defined as the discrete reduction of the hierarchy on $\mathcal{M}_{\rm red}$ is the same as the hierarchy obtained from the DS type reduction of the system on $\mathcal{M}^\gamma$ using the regular semisimple element $\Lambda \in \ell(\mathcal{G}^\gamma)$ and the gauge group $\mathcal{N}^{\rm T} = \exp\left(C^\infty\left(S^1, \mathcal{G}_{<0}^\gamma\right)\right)$.*

Proof. The statement follows by an elementrary "diagram chasing" argument. *Q.E.D.*

The commutativity of the diagram comprising the two DS type reductions and the respective discrete reductions does not depend on using the models of the DS-reduced systems provided by the respective lowest weight gauges, since the reduced systems have gauge independent meaning. One usually has other convenient gauges as well for describing KdV type systems and their "modified" versions. Another possibility which is often applicable is not to use any gauge at all for this purpose, but rather encode the gauge invariant information contained in the first order differential operator $\mathcal{L} \in \mathcal{M}_c$ in a corresponding higher order (pseudo-)differential operator. This will be illustrated by the examples in Subsection 5.2. In those examples the KdV system associated by DS reduction to a grade one regular semisimple element in the loop algebra of a classical Lie algebra, realized as $\mathcal{G}^\gamma$ for $\mathcal{G} = gl_n$, will turn out to be a discrete reduction of a hierarchy based on gl_n. In the above $\mathcal{G}$ was assumed to be a simple Lie algebra, but of course the whole construction applies to $\mathcal{G} = gl_n$ too.

5.2 Examples: Lax operators of Gelfand-Dickey type

A traditional method for describing KdV type systems that has proved fruitful in the past is to find a Gelfand-Dickey type model, where the gauge invariant dynamical variables of the system are encoded in a higher order (pseudo-)differential Lax operator L. The operator L is usually derived by an "elimination procedure" (see e.g. [1, 44, 17]) applied to the linear problem $\mathcal{L}\psi = 0$ for $\mathcal{L} \in \mathcal{M}_c$. The purpose of this subsection is to derive the Gelfand-Dickey type pseudo-differential Lax operators for a subset of the generalized KdV hierarchies resulting from the approach discussed in Subsection 5.1. We shall restrict ourselves to the cases for which $\mathcal{G}$ is a classical Lie algebra and the regular reductive subalgebra involved in the construction of the Heisenberg subalgebra of $\ell(\mathcal{G})$ contains only A or C type simple factors, see Table 4. The reason for this restriction is that the elimination procedure proves straightforwardly applicable in these cases. The cases involving the subalgebras D_{2p} with the conjugacy classes $(\bar{p}, \bar{p}) \subset \mathbf{W}(D_{2p})$ appear more difficult and are left aside for future work. It will turn out that the Lax operators obtained from the eliminitation procedure may be also derived by suitable restrictions from those related to gl_n, given in equations (1.3) and (1.4). The restriction consists in requiring the invariance of the Lax operator under some involutive discrete symmetry. Proposition 5.1 will be used to identify the Poisson brackets and the commuting Hamiltonians of the hierarchy in terms of the Gelfand-Dickey model. We shall study in some detail the C_n and B_n algebras, and essentially give the results for D_n.

Notations. Throughout this subsection, we use the 2×2 matrices σ, τ defined by

$$\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.17}$$

and the $p \times p$ matrices Y_p, η_p defined by

$$Y_p := \mathrm{diag}\left(\frac{p-1}{2}, \frac{p-3}{2}, \ldots, \frac{3-p}{2}, \frac{1-p}{2}\right), \qquad (\eta_p)_{ij} := \delta_{i,p+1-j} \quad \forall p > 1. \tag{5.18}$$

For a $p \times p$ matrix μ, $\tilde{\mu} := \eta_p \mu^t \eta_p$ is the transpose of μ with respect to the antidiagonal. As displayed also in (1.2), we have the regular semisimple element $\Lambda_p \in \ell(A_{p-1})$,

$$\Lambda_p := \lambda e_{p,1} + \sum_{i=1}^{p-1} e_{i,i+1}. \tag{5.19}$$

For any $p > 1$ and $s \in \mathbf{N}$, we fix some non-zero $d_i \in \mathbf{C}$ $(i = 1, \ldots, s)$ satisying $(d_i)^p \neq (d_k)^p$ for $i \neq k$ (compare with Table 1), and introduce the diagonal matrices

$$D_0 := \mathrm{diag}\,(d_1, \ldots, d_s)\,, \qquad D := \mathrm{diag}\left(D_0, -\tilde{D}_0\right), \qquad \Delta := -D^{-1}. \tag{5.20}$$

The $r \times r$ identity matrix is denoted by $\mathbf{1}_r$ for any integer $r > 1$. Finally, the adjoint $L^\dagger$ of some matrix pseudo-differential operator $L = \sum_{k \leq N} \alpha_k \partial^k$ is by definition $L^\dagger := \sum_{k \leq N} (-\partial)^k (\alpha_k)^t$.

- **Negative cycles in C_{ps}**

We first consider the algebra C_{ps} with the conjugacy class of $\mathbf{W}(C_{ps})$ associated to the signed partition $(\bar{p}, \ldots, \bar{p})$. This conjugacy class corresponds to the regular semisimple subalgebra $(C_p + \cdots + C_p) \subset C_{ps}$ in Table 4. Following the scheme outlined in Subsection 4.1, we first introduce the $2p \times 2p$ symplectic matrix Ω_{2p},

$$\Omega_{2p} := \sigma \otimes \eta_p, \quad \text{that is} \quad (\Omega_{2p})_{ij} = \epsilon(i,j)\delta_{i,2p+1-j}, \quad \epsilon(i,j) = \begin{cases} 1 \text{ if } i < j \\ -1 \text{ if } i > j. \end{cases} \tag{5.21}$$

The $2ps \times 2ps$ symplectic matrix Ω used to define $C_{ps} \subset gl_{2ps}$ is given by

$$\Omega := \mathbf{1}_s \otimes \Omega_{2p} = \mathrm{diag}(\Omega_{2p}, \ldots, \Omega_{2p}). \tag{5.22}$$

According to (4.1) the grading operator is d_{2p,I_0} with the $2ps \times 2ps$ diagonal matrix

$$I_0 := \mathbf{1}_s \otimes Y_{2p} = \mathrm{diag}(Y_{2p}, \ldots, Y_{2p}). \tag{5.23}$$

We also need the grade one regular semisimple element $\Lambda^C_{2p} \in \ell(C_p)$ given by

$$\Lambda^C_{2p} := \lambda e_{2p,1} + \sum_{i=1}^{p} e_{i,i+1} - \sum_{i=p+1}^{2p-1} e_{i,i+1}. \tag{5.24}$$

A grade one regular semisimple element $\Lambda \in \ell(C_{ps})$ is then furnished by

$$\Lambda = D_0 \otimes \Lambda^C_{2p} = \mathrm{diag}\left(d_1 \Lambda^C_{2p}, \ldots, d_s \Lambda^C_{2p}\right). \tag{5.25}$$

Let us perform the change of basis that gives rise to the permutation P on the indices of the $2ps \times 2ps$ matrices,

$$P(2kp + i) := 2s(i-1) + k + 1, \quad 1 \leq i \leq 2p, \quad 0 \leq k \leq s-1. \tag{5.26}$$

This amounts to exchanging the factors in the tensor products above, i.e., in the new basis the symplectic matrix is written as $\Omega = \Omega_{2p} \otimes \mathbf{1}_s$, the grade one regular semisimple element reads $\Lambda = \Lambda^C_{2p} \otimes D_0$, and the grading matrix becomes $I_0 = Y_{2p} \otimes \mathbf{1}_s$. It will be convenient that the entries of I_0 are non-increasing along the diagonal.

Now we derive the Lax operator for the KdV system following from the DS reduction. For this we apply the definitions of the constrained manifold $\mathcal{M}_c$ in (5.7) and the gauge group $\mathcal{N}$ in (5.8) to the case at hand. We then consider the linear problem for $\mathcal{L} \in \mathcal{M}_c$, that is the equation

$$\mathcal{L}\psi = (\partial + j + \Lambda)\psi = 0. \tag{5.27}$$

Here $\psi = (\psi_1^t, \psi_2^t, \ldots, \psi_{2p}^t)^t$ is a $2ps$-vector and the ψ_i $(i = 1, \ldots, 2p)$ are s-vectors. Equation (5.27) is covariant with respect to $\mathcal{N}$ if we complement (5.8) with the transformation rule

$$e^f : \psi \mapsto e^f \psi, \qquad \forall e^f \in \mathcal{N}. \tag{5.28}$$

Notice that the transformation in (5.28) leaves the component ψ_1 invariant, because f is now given by a $2ps \times 2ps$ block-triangular matrix having $s \times s$ zero blocks on and above the diagonal.

It convenient to proceed by restricting $\mathcal{L} \in \mathcal{M}_c$ to the *block-diagonal gauge*, where j is defined to have the form

$$j = \mathrm{diag}(\theta_1, \ldots, \theta_{2p}), \tag{5.29}$$

with

$$\theta_i \in C^\infty(S^1, gl_s), \quad \theta_{2p+1-i} = -\theta_i^t, \quad \forall\, i = 1, \ldots, 2p. \tag{5.30}$$

Inserting j in (5.29) into (5.27) yields the system

$$\begin{aligned}
&(\partial + \theta_i)\psi_i + D_0\psi_{i+1} = 0, \quad i = 1, \ldots, p,\\
&(\partial + \theta_i)\psi_i - D_0\psi_{i+1} = 0, \quad i = p+1, \ldots, 2p-1,\\
&(\partial + \theta_{2p})\psi_{2p} + \lambda D_0 \psi_1 = 0.
\end{aligned} \tag{5.31}$$

Upon elimination, this system leads to the eigenvalue equation

$$L\psi_1 = \lambda\psi_1, \tag{5.32}$$

where L is the $s \times s$ matrix differential operator of order $2p$ given by

$$L = (-1)^{p+1} D_0^{-1}(\partial + \theta_{2p}) D_0^{-1}(\partial + \theta_{2p-1}) \cdots D_0^{-1}(\partial + \theta_1). \tag{5.33}$$

As a consequence of (5.30), L is invariant with respect to the operation

$$L \mapsto \hat{L} := D_0^{-1} L^\dagger D_0. \tag{5.34}$$

If we use an expanded form of the Lax operator L, we have

$$L = (-1)^{p+1} D_0^{-2p}\partial^{2p} + D_0^{-1}\sum_{k=1}^{2p}(u_k\partial^{2p-k} + \partial^{2p-k}u_k), \tag{5.35}$$

where the KdV fields $u_k \in C^\infty(S^1, gl_s)$ satisfy $u_k^t = (-1)^k u_k$ by the invariance property $\hat{L} = L$.

Since the above elimination procedure can be turned backwards, equation (5.32) encodes all gauge invariant information contained in the original linear problem (5.27). It is easy to see that the KdV fields u_k in (5.35) are related by an invertible differential polynomial substitution to the entries of the gauge fixed current in the lowest weight gauge of (5.9). The fields θ_i in (5.33) are the dynamical variables of a "modified" version of the KdV hierarchy. Expanding the factorized operator (5.33) yields a generalization of the well-known Miura map.

The KdV system having the Lax operator L in (5.35) may be interpreted as a discrete reduction (in the sense of Subsection 5.1) of a KdV system based on gl_n for $n = 2ps$. In fact, the subalgebra C_{ps} of gl_{2ps} is the fixed point set of the involution $\gamma : gl_{2ps} \to gl_{2ps}$ defined by

$$\gamma : X \mapsto \gamma(X) := -\Omega^{-1} X^t \Omega \qquad \forall\, X \in gl_{2ps}, \tag{5.36}$$

and the element $\Lambda \in \ell(C_{ps}) \subset \ell(gl_{2ps})$ given in (5.25) is also a grade one regular semisimple element of $\ell(gl_{2ps})$ (and of $\ell(A_{2ps-1})$). From this point of view Λ is associated to the partition $(2p, \ldots, 2p)$ of $n = 2ps$ representing a regular conjugacy class in $\mathbf{W}(A_{2ps-1})$. Performing the DS reduction using gl_{2ps} instead of C_{ps} leads to a KdV system whose Lax operator has the form in (5.35), but with *arbitrary* $u_k \in C^\infty(S^1, gl_s)$. The related modified KdV system is given

by the operator (5.33) with unrestricted $\theta_i \in C^\infty(S^1, gl_s)$. Proposition 5.1 and what is known about the gl_n case [17] enables us to give a more detailed description of the present generalized KdV hierarchy in the Gelfand-Dickey framework. We next explain this in detail.

Let M be the manifold of Lax operators L of the form in (5.35) with arbitrary KdV fields $u_k \in C^\infty(S^1, gl_s)$. Recall from [17] that the compatible PBs on M, regarded as a model of the DS-reduced space $\mathcal{M}_{\rm red}$ associated to gl_{2ps}, are the standard first and second matrix Gelfand-Dickey PBs [2, 3, 4] defined respectively by

$$\{f_A, f_B\}^{(1)}(L) = \mathrm{Tr}\left(L\left([A_+, B_+] - [A_-, B_-]\right)\right), \tag{5.37}$$

$$\{f_A, f_B\}^{(2)}(L) = \mathrm{Tr}\left(BL(AL)_+ - B(LA)_+L\right). \tag{5.38}$$

Here Tr is the Adler trace [4] of matrix pseudo-differential operators (PDOs) given by

$$\mathrm{Tr}(A) := \int_{S^1} \mathrm{tr}\,\mathrm{res}\,(A)\,, \quad \mathrm{res}\,(A) := A_{-1} \qquad \forall\, A = \sum_{k\le k_0} A_k \partial^k, \quad A_k \in C^\infty(S^1, gl_s). \tag{5.39}$$

For an arbitrary PDO A, we use the splitting $A = A_+ + A_-$ into parts containing non-negative and negative powers of ∂, respectively. In formulas (5.37), (5.38) f_A is the linear function on M defined by $f_A(L) := \mathrm{Tr}\,(AL)$ for any fixed $s\times s$ matrix PDO A.

We have the discrete symmetry given by the Poisson mapping

$$\hat\gamma : M \to M, \qquad \hat\gamma(L) := \hat L = D_0^{-1} L^\dagger D_0, \quad \forall L \in M. \tag{5.40}$$

The symmetry $\hat\gamma$ is induced from the action (5.13) of γ in (5.36) on the constrained manifold of the DS reduction considered for gl_{2ps}. This is easily seen with the aid of the corresponding block-diagonal gauge, whose gauge section is mapped to itself by γ. The phase space of the "discrete reduced" hierarchy is the fixed point set $M^{\hat\gamma} \subset M$ of $\hat\gamma$. Proposition 5.1 implies that the induced PBs on the fixed point set $M^{\hat\gamma}$, which is a model of $\mathcal{M}^\gamma_{\rm red}$, are given by formulas (5.37) and (5.38), where A and B have to be restricted to PDOs that are anti-symmetric with respect to the transformation $\hat\gamma$. Indeed, if $\hat\gamma(A) := D_0^{-1}A^\dagger D_0 = -A$, then $f_A(\hat\gamma(L)) = f_A(L)$.

The commuting Hamiltonians of the hierarchy on M induced by the DS reduction may be obtained as follows [17]. First one has to diagonalize $L \in M$ in the algebra of PDOs, i.e., for any L one has to determine a diagonal PDO L_d,

$$L_d = (-1)^{p+1} D_0^{-2p}\partial^{2p} + \sum_{k=1}^\infty a_k \partial^{2p-k}, \quad \text{with} \quad a_k \quad \text{diagonal matrix} \quad \forall k, \tag{5.41}$$

for which

$$L = gL_d g^{-1}, \qquad g = \mathbf{1}_s + \sum_{k=1}^\infty g_k \partial^{-k}, \quad \text{with} \quad g_k \quad \text{off-diagonal matrix} \quad \forall k. \tag{5.42}$$

By (5.41), (5.42), $L_d(L)$ and $g(L)$ are uniquely determined (differential polynomial) functions of $L\in M$. The commuting Hamiltonians are then provided by

$$H_{0,i}(L) := \int_{S^1} (u_1)_{ii}\,, \qquad \forall\, i = 1, \dots, s, \tag{5.43}$$

$$H_{k,i}(L) := \int_{S^1} \mathrm{res}\ (L_d(L))_{ii}^{k/2p}, \qquad \forall\, i = 1, \ldots, s, \quad k = 1, 2, \ldots, \tag{5.44}$$

where $(L_d(L))^{1/2p}$ is a fixed $2p$th root of $L_d(L)$. Thanks to the uniqueness property of the diagonalization procedure in (5.41), (5.42) and the identity $\mathrm{Tr}\left(A^\dagger\right) = -\mathrm{Tr}\,(A)$, we can verify

$$H_{k,i}\left(\hat{\gamma}\,(L)\right) = (-1)^{k+1} H_{k,i}(L), \qquad \forall\, i = 1, \ldots, s, \quad k = 0, 1, \ldots. \tag{5.45}$$

According to Proposition 5.1, the commuting Hamiltonians of the discrete reduced hierarchy on $M^{\hat{\gamma}} \simeq \mathcal{M}^{\gamma}_{\mathrm{red}}$ are furnished by the restrictions of the $\hat{\gamma}$-invariant Hamiltonians on $M \simeq \mathcal{M}_{\mathrm{red}}$. We see from (5.45) that the invariant Hamiltonians are now the $H_{k,i}(L)$ for k any *odd* natural number. This completes our description of the PDO model of the generalized KdV hierarchy following from DS reduction in the case $(\bar{p}, \ldots, \bar{p}) \subset \mathbf{W}(C_{ps})$. The result is analogous to the $s = 1$ "scalar case", for which the C_p-type DS hierarchy is the self-adjoint reduction of the gl_{2p}-type Gelfand-Dickey (n-KdV for $n = 2p$) hierarchy [1].

- **Positive cycles in** C_{ps}

We now turn to the case of positive cycles of odd length, $(p, \ldots, p)$ with $p = 2q + 1$, in C_{ps}. The regular semisimple subalgebra associated in Table 4 to this conjugacy class of $\mathbf{W}(C_{ps})$ is $(A_{p-1} + \cdots + A_{p-1}) \subset C_{ps}$ The symplectic matrix Ω is still given by (5.22). The grading of $\ell(C_{ps})$ is now defined by the operator d_{p,I_0} with $I_0 := \mathbf{1}_{2s} \otimes Y_p$. Using (5.17)–(5.20), the grade one regular semisimple element $\Lambda \in \ell(C_{ps})$ is given as $\Lambda = D_0 \otimes \tau \otimes \Lambda_p$.

Let us perform the permutation

$$\begin{aligned} P(2kp + i) &:= 2s(i-1) + k + 1, \\ P(2kp + p + i) &:= 2si - k, \end{aligned} \qquad 1 \le i \le p, \quad 0 \le k \le s-1. \tag{5.46}$$

After this permutation, the symplectic matrix writes as $\Omega = \eta_p \otimes \Omega_{2s}$ and the grading matrix becomes $I_0 = Y_p \otimes \mathbf{1}_{2s}$, which has non-increasing entries along the diagonal. Finally, with D given in (5.20), we have

$$\Lambda = \Lambda_p \otimes D. \tag{5.47}$$

Like in the previous case, we consider the linear problem (5.27). Now the $2ps$-vector ψ is decomposed as $\psi = (\psi_1^t, \ldots, \psi_p^t)^t$ in terms of the $2s$-vectors ψ_i for $i = 1, \ldots, p$. In the block-diagonal gauge j has the form

$$j = \mathrm{diag}(\theta_1, \ldots, \theta_p), \tag{5.48}$$

with

$$\theta_i \in C^\infty\left(S^1, gl_{2s}\right), \qquad \theta_i = -\Omega_{2s}\theta^t_{p+1-i}\Omega_{2s}^{-1} \qquad \forall\, i = 1, \ldots, p. \tag{5.49}$$

Combining (5.27) with (5.47), (5.48), we obtain the system

$$\begin{aligned} &(\partial + \theta_i)\psi_i + D\psi_{i+1} = 0, \quad 1 \le i \le p-1, \\ &(\partial + \theta_p)\psi_p + \lambda D\psi_1 = 0. \end{aligned} \tag{5.50}$$

By elimination, we then get the eigenvalue equation $L\psi_1 = \lambda\psi_1$, where the $2s \times 2s$ matrix Lax operator L is given by

$$L = \Delta(\partial + \theta_p) \cdots \Delta(\partial + \theta_1), \tag{5.51}$$

with Δ defined in (5.20). On account of (5.49) and $\Omega_{2s}\Delta^t\Omega_{2s}^{-1} = -\Delta$, L in (5.51) is invariant with respect to the transformation

$$L \mapsto \hat{L} := \Delta\Omega_{2s}L^\dagger\Omega_{2s}^{-1}\Delta^{-1}. \tag{5.52}$$

If we write the Lax operator in expanded form as

$$L = \Delta^p\partial^p + \Delta\sum_{k=1}^{p}(u_k\partial^{p-k} + \partial^{p-k}u_k), \tag{5.53}$$

then the invariance property $L = \hat{L}$ yields $u_k = (-1)^k\Omega_{2s}u_k^t\Omega_{2s}^{-1}$.

Similarly to the previous example, we see that the KdV system possessing the Lax operator in (5.53) is a discrete reduction of a system of the type in (1.3), which is based on gl_n with the partition $(p,\ldots,p)$ of $n = 2ps$. It follows that the compatible PBs of the KdV system obtained from the DS reduction are given by (5.37), (5.38), where A and B have to be restricted to PDOs that are anti-symmetric with respect to the discrete symmetry in (5.52),

$$\Delta\Omega_{2s}A^\dagger\Omega_{2s}^{-1}\Delta^{-1} = -A, \qquad \Delta\Omega_{2s}B^\dagger\Omega_{2s}^{-1}\Delta^{-1} = -B. \tag{5.54}$$

Before the discrete reduction, i.e., on the space of Lax operators of the form in (5.53) but with arbitrary coefficients $u_k \in C^\infty(S^1, gl_{2ps})$, the commuting Hamiltonians are $H_{0,i}(L)$ defined like in (5.43) and $H_{k,i}(L)$ defined by

$$H_{k,i}(L) := \int_{S^1} \mathrm{res}\ (L_d(L))_{ii}^{k/p}, \qquad \forall\, i = 1,\ldots,2s, \quad k = 1,2,\ldots. \tag{5.55}$$

Here $(L_d(L))^{1/p}$ is a fixed pth root of the diagonal PDO $L_d(L)$ determined analogously to (5.42). Choosing the leading term of $(L_d(L))^{1/p}$ to be $\Delta\partial$, we find the transformation property

$$H_{k,i}(\hat{L}) = -H_{k,2s+1-i}(L), \qquad \forall\, i = 1,\ldots,2s, \quad k = 0,1,\ldots. \tag{5.56}$$

Therefore the Hamiltonians of the KdV system based on gl_{2ps} that are invariant with respect to the discrete symmetry in (5.52) are furnished by

$$H_{k,i}^+(L) := H_{k,i}(L) - H_{k,2s+1-i}(L), \qquad \forall\, i = 1,\ldots,s, \quad k = 0,1,\ldots. \tag{5.57}$$

As a consequence of Proposition 5.1, the Hamiltonians obtained by inserting the Lax operator L in (5.53) into (5.57) coincide with those resulting from "abelianization" in the DS reduction realization of the generalized KdV system associated to $(p,\ldots,p) \subset \mathbf{W}(C_{ps})$.

- **Positive cycles in D_{ps}**

The case of positive cycles of odd length, $(p,\ldots,p)$ with $p = 2q+1$, in $\mathbf{W}(D_{ps})$ is very similar. We end up with a Lax operator L that has the factorized form in (5.51), where the matrices θ_i now satisfy $\theta_i = -\tilde{\theta}_{p+1-i}$. Thus the invariance property of L is

$$\hat{L} = L \qquad \text{for} \qquad L \mapsto \hat{L} := \Delta\eta_{2s}L^\dagger\eta_{2s}\Delta^{-1}. \tag{5.58}$$

The expanded form of the Lax operator can be written as in (5.53), where the $2s \times 2s$ matrix KdV fields u_k are now subject to $u_k = (-1)^k\tilde{u}_k$. This KdV system is another discrete reduction of the system based on gl_n with the partition $(p,\ldots,p)$ of $n = 2ps$. The PBs of this system following from the DS reduction can be obtained from the Gelfand-Dickey PBs in (5.37), (5.38) by restricting A and B to be anti-symmetric PDOs with respect to the transformation in (5.58). The commuting Hamiltonians can be characterized analogously to the preceding example.

• **Positive cycles in** B_{ps}

Now we deal with the case of positive cycles of odd length $(p, \ldots, p) \subset \mathbf{W}(B_{ps})$, $p = 2q+1$. The corresponding regular semisimple subalgebra is given by $(A_{p-1} + \cdots + A_{p-1}) \subset B_{ps}$. The $(2ps+1) \times (2ps+1)$ matrix η defining the B_{ps}-invariant symmetric form can be taken to be

$$\eta := \begin{pmatrix} \mathbf{1}_s \otimes \eta_{2p} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.59}$$

The grading of $\ell(B_{ps})$ is defined by the operator d_{p,I_0} with

$$I_0 := \begin{pmatrix} \mathbf{1}_{2s} \otimes Y_p & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.60}$$

The relevant grade one regular semisimple element $\Lambda \in \ell(B_{ps})$ can be written as

$$\Lambda = \begin{pmatrix} D_0 \otimes \tau \otimes \Lambda_p & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.61}$$

For the notations, see (5.17)–(5.20).

Let us change the basis using P in (5.46) to permute the first $2ps$ indices together the prescription $P(2ps+1) := 2ps+1$ for the last index. The matrix of the symmetric form left invariant by $B_{ps} \subset gl_{2ps+1}$ then becomes

$$\eta = \begin{pmatrix} \eta_{2ps} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.62}$$

The grading matrix reads

$$I_0 = \begin{pmatrix} Y_p \otimes \mathbf{1}_{2s} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.63}$$

The grade one regular element takes the form

$$\Lambda = \begin{pmatrix} \Lambda_p \otimes D & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.64}$$

In the linear problem (5.27) the vector ψ may be now decomposed as $\psi = (\psi_1^t, \ldots, \psi_p^t, \phi)^t$, where the ψ_i $(i = 1, \ldots, p)$ are $2s$-vectors and ϕ is the last component of ψ. We now define the "block-diagonal" gauge by restricting the $(2s+1)\times(2s+1)$ matrix valued field $j \in C^\infty(S^1, B_{ps})$ in $\mathcal{L} = (\partial + j + \Lambda) \in \mathcal{M}_c$ to have the form

$$j = \begin{pmatrix} \theta_1 & & & & & \\ & \ddots & & & & \\ & & \theta_{q+1} & & & b \\ & & & \ddots & & \\ & & & & \theta_p & \\ & & c^t & & & 0 \end{pmatrix}. \tag{5.65}$$

The non-vanishing entries of j in (5.65) have grade zero with respect to I_0 in (5.63) and satisfy

$$\theta_i \in C^\infty(gl_{2s}, S^1), \quad \theta_i = -\tilde{\theta}_{p+1-i} \quad \forall i = 1, \ldots, p, \qquad b, c \in C^\infty(S^1, \mathbf{C}^{2s}), \quad c = -\eta_{2s} b. \tag{5.66}$$

Substituting (5.64), (5.65) into (5.27), we obtain the system

$$\begin{aligned}
&(\partial + \theta_i)\psi_i + D\psi_{i+1} = 0, \quad i = 1, \ldots, q,\, q+2, \ldots, 2q, \\
&(\partial + \theta_{q+1})\psi_{q+1} + D\psi_{q+2} + b\phi = 0, \\
&(\partial + \theta_p)\psi_p + \lambda D\psi_1 = 0, \\
&\partial\phi + c^t\psi_{q+1} = 0.
\end{aligned} \tag{5.67}$$

The component ϕ may be eliminated using the last equation, which yields

$$\phi = -\partial^{-1} c^t \psi_{q+1}. \tag{5.68}$$

Plugging (5.68) back into (5.67), further elimination leads to the eigenvalue equation

$$L\psi_1 = \lambda\psi_1, \tag{5.69}$$

where L is the following $2s \times 2s$ matrix *pseudo-differential* operator:

$$L = \Delta(\partial + \theta_p) \cdots \Delta(\partial + \theta_{q+2})\Delta \left[\partial + \theta_{q+1} - b\partial^{-1}c^t\right] \Delta(\partial + \theta_q) \cdots \Delta(\partial + \theta_1). \tag{5.70}$$

Because of (5.66), L in (5.70) has the invariance property

$$\hat{L} = L \qquad \text{for} \qquad L \mapsto \hat{L} := \Delta\eta_{2s}L^\dagger\eta_{2s}\Delta^{-1}, \qquad (\Delta = -D^{-1}). \tag{5.71}$$

The Lax operator given by (5.70) can be written in expanded form as

$$L = \Delta^p\partial^p + \Delta\sum_{k=1}^{p}(u_k\partial^{p-k} + \partial^{p-k}u_k) - \Delta z_+\partial^{-1}z_-^t, \tag{5.72}$$

$$u_k \in C^\infty(S^1, gl_{2s}), \quad u_k = (-1)^k\tilde{u}_k \quad \forall k = 1, \ldots, p, \quad z_+, z_- \in C^\infty(S^1, \mathbf{C}^{2s}), \quad z_- = -\eta_{2s}z_+. \tag{5.73}$$

The above Lax operator can be also derived by performing the elimination on the linear problem (5.27) in a DS gauge. For this it is convenient to consider the gauge section $\mathcal{M}_{\rm DS} \subset \mathcal{M}_c$ which by definition consists of the first order differential operators $\mathcal{L} = (\partial + j_{\rm DS} + \Lambda)$ with

$$j_{\rm DS} := \begin{pmatrix}
v_1 & & & & & \\
v_2 & & & & & \\
\vdots & & & & & \\
v_{p-1} & & & & & \\
v_p & -\tilde{v}_{p-1} & \cdots & -\tilde{v}_2 & -\tilde{v}_1 & z_+ \\
z_-^t & & & & &
\end{pmatrix}, \tag{5.74}$$

where $v_k \in C^\infty(S^1, gl_{2s})$ subject to $v_k = (-1)^k\tilde{v}_k$, and $z_\pm$ are given in (5.73). The gauge section $\mathcal{M}_{\rm DS}$ is a one-to-one model of the reduced space $\mathcal{M}_{\rm red} = \mathcal{M}_c/\mathcal{N}$ following from the

DS reduction in the present case. The fields v_k in (5.74) and the u_k in (5.72) are related by an invertible differential polynomial substitution, but the field z_- appears only in quadratic combinations in the expression (5.72) of L. This means that the manifold of Lax operators L in (5.72) is now *not* a one-to-one model of the space $\mathcal{M}_{\rm red}$. A convenient parametrization of $\mathcal{M}_{\rm red}$ is furnished by the set of all pairs (L_+, z_-), where L_+ is the differential operator part of L in (5.72) and $z_- \in C^\infty(S^1, \mathbf{C}^{2s})$. This is somewhat similar to the situation found in [1] for the principal case of the D_n algebras, for which the Lax oparators are skew-symmetric scalar pseudo-differential operators having a negative part of the form $z\partial^{-1}z$ with $z \in C^\infty(S^1, \mathbf{C})$.

Finally, we note that the above KdV system associated by DS reduction to the conjugacy class $(p, \ldots, p) \subset \mathbf{W}(B_{ps})$ can be viewed as a discrete reduction of a KdV system based on gl_{2ps+1} with the corresponding partition $(p, \ldots, p, 1)$, where $p = 2q + 1$ occurs $2s$ times. The phase space of the system based on gl_{2ps+1} consists of the quadruples (L_+, y_+, y_-, w) appearing in (1.4). The PBs and the commuting Hamiltonians are described in these variables in [18].

- **Positive cycles plus a 1-cycle in** D_{ps+1}

The case of positive cycles of odd length $p = 2q+1$ plus a 1-cycle $(p, \ldots, p, 1) \subset \mathbf{W}(D_{ps+1})$ resembles the last one. Without entering into details, let us give the form of the $(2ps+2) \times (2ps+2)$ matrix valued field j in the "block-diagonal" gauge,

$$j = \begin{pmatrix} \theta_1 & & & & & \\ & \ddots & & & & \\ & & \theta_{q+1} & & & b \\ & & & \ddots & & \\ & & & & \theta_p & \\ & & c^t & & & d \end{pmatrix}. \tag{5.75}$$

Here the θ_i $(i = 1, \ldots, p)$ are $2s \times 2s$ matrices satisfying $\theta_i = -\tilde{\theta}_{p+1-i}$, b and c are rectangular $2s \times 2$ matrices related by $c = -\eta_{2s} b \eta_2$, and d is a 2×2 matrix constrained by $d = -\tilde{d}$. The corresponding pseudo-differential Lax operator L is given in factorized form as

$$L = \Delta(\partial + \theta_p) \cdots \Delta(\partial + \theta_{q+2})\Delta\left[\partial + \theta_{q+1} - b(\partial + d)^{-1}c^t\right]\Delta(\partial + \theta_q) \cdots \Delta(\partial + \theta_1), \tag{5.76}$$

with Δ in (5.20). The operator L in (5.76) enjoys the invariance property (5.71) and can be expanded as

$$L = \Delta^p \partial^p + \Delta \sum_{k=1}^{p} (u_k \partial^{p-k} + \partial^{p-k} u_k) - \Delta z_+ (\partial + d)^{-1} z_-^t, \tag{5.77}$$

where the $2s \times 2s$ matrices u_k satisfy $u_k = (-1)^k \tilde{u}_k$ and the rectangular $2s \times 2$ matrices z_+ and z_- are related by $z_- = -\eta_{2s} z_+ \eta_2$.

The generalized KdV system at hand is related to a system based on $\mathcal{G} := gl_n$ with the partition $(p, \ldots, p, 1, 1)$ of $n = 2ps + 2$ by means of an involution $\gamma : \mathcal{G} \to \mathcal{G}$ for which $\mathcal{G}^\gamma = D_{ps+1}$. If there are more than one extra 1-cycles contained in the partition n, then graded *regular* semisimple elements do not exist in the corresponding Heisenberg subalgebra of $\ell(gl_n)$. However, in the cases $(p, \ldots, p, 1, \ldots, 1)$ — with an arbitrary number of 1-cycles — the DS reduction still goes through without any difficulty using a grade one semisimple element from the Heisenberg subalgebra. The resulting KdV type hierarchies are studied in [18].

6 Some remarks on non-abelian Toda systems

In the preceding section we associated generalized KdV systems to grade one regular semisimple elements of $\ell(\mathcal{G})$. For completeness, below we wish to present the well-known definition of corresponding "non-abelian affine Toda" systems, and work out an example.

To obtain a non-abelian[4] affine Toda model, consider a grade 1 and a grade -1 regular semisimple element, Λ and $\bar{\Lambda}$, from some non-principal Heisenberg subalgebra of $\ell(\mathcal{G})$. The grading is given by the operator d_{m,I_0} in (4.1). For simplicity we here assume that $\mathrm{ad} I_0$ has only *integral* eigenvalues. Similarly to equations (5.1), (5.2), for Λ and $\bar{\Lambda}$ given by

$$\Lambda = I_+ + \lambda C_-, \qquad \bar{\Lambda} = \bar{I}_- + \lambda^{-1}\bar{C}_+, \tag{6.1}$$

we suppose that

$$[C_-, \mathcal{G}_{<0}] = \{0\}, \qquad [\bar{C}_+, \mathcal{G}_{>0}] = \{0\}. \tag{6.2}$$

The non-abelian affine Toda equation is a relativistically invariant field equation for a field $g(x,t)$ that varies in a connected (non-abelian) Lie group G_0 generated by the grade zero Lie subalgebra $\mathcal{G}_0 \subset \mathcal{G}$. It is postulated to be the zero curvature equation

$$[\mathcal{L}_+, \mathcal{L}_-] = 0, \tag{6.3}$$

with

$$\mathcal{L}_+ := \partial_+ + g^{-1}\partial_+ g + \Lambda, \qquad \mathcal{L}_- := \partial_- + g^{-1}\bar{\Lambda} g, \tag{6.4}$$

where $\partial_\pm := (\partial_x \pm \partial_t)$. More explicitly, the field equation (6.3) reads

$$\partial_-(g^{-1}\partial_+ g) = [I_+, g^{-1}\bar{I}_- g] + [C_-, g^{-1}\bar{C}_+ g]. \tag{6.5}$$

This is a deformation of the non-abelian conformal Toda equation obtained from (6.5) by omitting the second term on the right hand side. The model admits two infinite series of conserved local currents, which may be obtained with the aid of the abelianization of $\mathcal{L}_x = \mathcal{L}_+ - \mathcal{L}_-$ and that of $\tilde{\mathcal{L}}_x := \tilde{\mathcal{L}}_+ - \tilde{\mathcal{L}}_-$, respectively, where the operators

$$\tilde{\mathcal{L}}_+ := \partial_+ + g\Lambda g^{-1}, \qquad \tilde{\mathcal{L}}_- := \partial_- - \partial_- g g^{-1} + \bar{\Lambda} \tag{6.6}$$

enter the alternative zero curvature representation

$$[\tilde{\mathcal{L}}_+, \tilde{\mathcal{L}}_-] = 0 \tag{6.7}$$

of the field equation (6.5).

The models defined by (6.1)–(6.4) are special cases of those proposed by Leznov and Saveliev in [8]. They are distinguished by the applicability of the abelianization procedure described in (1.9),(1.10). It is well-known [1, 7, 9, 10, 11, 12] that infinitely many conserved local currents exist also in the non-abelian affine Toda models associated to grade ± 1 semisimple, not necessarily regular elements from $\ell(\mathcal{G})$. In general the conserved local currents are labelled by the basis elements of the *centre of the centralizer* of Λ ($\bar{\Lambda}$) with non-positive (non-negative) grades.

[4] The *abelian* affine Toda model is related to the principal Heisenberg subalgebra as is well-known.

Suppose that we consider a regular conjugacy class of the Weyl group that has the product structure in (4.4). The corresponding Toda model then will have the interpretation as a "coupled system" containing the Toda systems associated to grade ± 1 regular elements from the primitive Heisenberg subalgebras $\widetilde{\mathcal{H}}_{k,\hat{w}_k} \subset \ell(\mathcal{G}_k)$ for $k = 1, \dots, r$ (see (4.4)–(4.10)), which are coupled together by means of certain extra fields. The extra fields correspond to the part of $\mathcal{G}_0$ outside the regular semisimple subalgebra given in (4.5). It is easy to see that the extra fields can be consistently set to zero in the field equation (6.5), which then reduces to a decoupled set of Toda equations associated to the primitive conjugacy classes $[w_k] \subset \mathbf{W}(\mathcal{G}_k)$.

We now wish to elaborate the non-abelian affine Toda equation (6.5) for the two negative cycles case $(\bar{p}, \bar{p})$ in D_{2p} for any $p \geq 2$. The motivation for considering this series of examples is that for the classical Lie algebras $(\bar{p}, \bar{p}) \subset \mathbf{W}(D_{2p})$ are the only conjugacy classes of the Weyl group which are regular, primitive and different from a Coxeter class. Choosing all constants a_i, b_i in (4.27) to be 1 for simplicity, the grade 1 generators of the corresponding Heisenberg subalgebra are

$$\begin{aligned}
\Lambda_{1,1} &= \lambda(e_{2p,1} - e_{2p+1,2}) + \sum_{k=1}^{p} e_{k,k+1} - \sum_{k=1}^{p} e_{p+k,p+k+1}, \\
\Lambda_{1,2} &= \lambda(e_{4p,1} - e_{2p+1,2p+2}) + (e_{4p,2p+1} - e_{1,2p+2}) + \sum_{k=1}^{p-1} e_{2p+1+k,2p+2+k} - \sum_{k=1}^{p-1} e_{3p+k,3p+k+1},
\end{aligned} \tag{6.8}$$

and the grade -1 generators, $\Lambda_{-1,i} \sim \lambda^{-1}(\Lambda_{1,i})^{2p-1}$, are

$$\Lambda_{-1,1} = \lambda^{-1}(e_{1,2p} - e_{2,2p+1}) + (e_{2,1} - e_{2p+1,2p}) + 2\sum_{k=1}^{p-1} e_{2+k,1+k} - 2\sum_{k=1}^{p-1} e_{p+1+k,p+k}, \tag{6.9}$$

$$\Lambda_{-1,2} = \lambda^{-1}(e_{1,4p} - e_{2p+2,2p+1}) + (e_{2p+1,4p} - e_{2p+2,1}) + 2\sum_{k=1}^{p-1} e_{2p+2+k,2p+1+k} - 2\sum_{k=1}^{p-1} e_{3p+1+k,3p+k}.$$

These formulas are valid in the basis where the symmetric form η and the grading K are given by (4.20) and (4.21), and it is convenient to permute the basis so that in the new basis they take the following block-form:

$$\begin{aligned}
K &= \mathrm{diag}\,(p, (p-1)\mathbf{1}_2, \dots, -(p-1)\mathbf{1}_2, -p)\,, \\
\eta &= \mathrm{antidiag}\,(1, \mathbf{1}_2, \dots, \mathbf{1}_2, 1)\,.
\end{aligned} \tag{6.10}$$

According to the grading defined by K, we can write all matrices in a $(2p+1) \times (2p+1)$ block-form, with the various blocks being 2×2 matrices and 2-component column or row vectors, respectively. In order to write down the grade ± 1 regular elements $\Lambda = d_1\Lambda_{1,1} + d_2\Lambda_{2,1}$ and $\bar{\Lambda} := \bar{d}_1\Lambda_{-1,1} + \bar{d}_2\Lambda_{-1,2}$, it is useful to introduce

$$\alpha := \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad \beta := \begin{pmatrix} d_1 \\ -d_2 \end{pmatrix}, \quad D_0 := \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \tag{6.11}$$

and

$$\bar{\alpha} := \begin{pmatrix} \bar{d}_1 \\ \bar{d}_2 \end{pmatrix}, \quad \bar{\beta} := \begin{pmatrix} \bar{d}_1 \\ -\bar{d}_2 \end{pmatrix}, \quad \bar{D}_0 := 2\begin{pmatrix} \bar{d}_1 & 0 \\ 0 & \bar{d}_2 \end{pmatrix}. \tag{6.12}$$

Using this notation, in the new basis we have

$$\Lambda = e_{1,2} \otimes \beta^t + \sum_{k=2}^{p} e_{k,k+1} \otimes D_0 - \sum_{k=p+1}^{2p} e_{k,k+1} \otimes D_0 - e_{2p,2p+1} \otimes \beta \\ + \lambda \left(e_{2p,1} \otimes \alpha - e_{2p+1,2} \otimes \alpha^t \right), \tag{6.13}$$

and

$$\bar{\Lambda} = e_{2,1} \otimes \bar{\beta} + \sum_{k=2}^{p} e_{k+1,k} \otimes \bar{D}_0 - \sum_{k=p+1}^{2p} e_{k+1,k} \otimes \bar{D}_0 - e_{2p+1,2p} \otimes \bar{\beta}^t \\ + \lambda^{-1} \left(e_{1,2p} \otimes \bar{\alpha}^t - e_{2,2p+1} \otimes \bar{\alpha} \right). \tag{6.14}$$

We write the group element $g \in G_0$ in the block-diagonal form

$$g = \sum_{k=1}^{2p+1} e_{k,k} \otimes g_k, \tag{6.15}$$

where $g_1, g_{2p+1} \in GL(1)$ and $g_k \in GL(2)$ otherwise, with the condition $g^t \eta g = \eta$ translating into

$$g_{2p+2-l} = (g_l^{-1})^t \quad \text{for} \quad l = 1, \ldots, p+1. \tag{6.16}$$

Then the non-abelian affine Toda equation (6.5) takes the form,

$$\begin{aligned}
\partial_-(g_1^{-1}\partial_+ g_1) &= \beta^t g_2^{-1} \bar{\beta} g_1 - g_1^{-1} \bar{\alpha}^t (g_2^t)^{-1} \alpha, \\
\partial_-(g_2^{-1}\partial_+ g_2) &= D_0 g_3^{-1} \bar{D}_0 g_2 - g_2^{-1} \bar{\beta} g_1 \beta^t - g_2^{-1} \bar{\alpha} g_1^{-1} \alpha^t, \\
\partial_-(g_k^{-1}\partial_+ g_k) &= D_0 g_{k+1}^{-1} \bar{D}_0 g_k - g_k^{-1} \bar{D}_0 g_{k-1} D_0, \qquad 2 < k \leq p+1,
\end{aligned} \tag{6.17}$$

where $g_{p+2}^{-1} = g_p^t$. The conformal Toda equation corresponding to equation (6.17) can be obtained by dropping the terms containing α and $\bar{\alpha}$. The simplest version of equation (6.17) arises for the Lie algebra D_4, and describes a $GL(2)$ valued field g_2 interacting with two "scalars" $g_1 \in GL(1)$ and $g_3 \in O(2)$.

7 Conclusion

In this paper we studied a class of generalized KdV hierarchies associated by Drinfeld-Sokolov reduction to regular semisimple elements of grade one in the non-twisted loop algebras. We made use of the fact that the classification of the graded regular semisimple elements in a loop algebra $\ell(\mathcal{G})$ can be reduced to the known [27] classification of the regular conjugacy classes in the Weyl group $\mathbf{W}(\mathcal{G})$ of the underlying simple Lie algebra $\mathcal{G}$. The regular conjugacy classes in $\mathbf{W}(\mathcal{G})$ parametrize the non-equivalent Heisenberg subalgebras of $\ell(\mathcal{G})$ containing graded regular semisimple elements. Restricting our attention to the *classical* simple Lie algebras, we exhibited a relationship between the regular conjugacy classes in $\mathbf{W}(\mathcal{G})$ and certain sl_2 subalgebras of $\mathcal{G}$.

Let $[w] \subset \mathbf{W}(\mathcal{G})$ be a regular conjugacy class of order m for $\mathcal{G}$ a classical simple Lie algebra. We have seen that there exists a lift $\hat{w}$ of a representative $w \in [w]$ that takes the form $\hat{w} = \exp\left(2i\pi \mathrm{ad} I_0/m\right)$ in such a way that I_0 is the semisimple element ("defining vector") of an sl_2 subalgebra of $\mathcal{G}$ for which the largest eigenvalue of $\mathrm{ad}I_0$ is $(m-1)$. Any regular element Λ of minimal positive grade from the corresponding Heisenberg subalgebra has the form $\Lambda = (C_1 + \lambda C_{-(m-1)})$, where $[I_0, C_k] = kC_k$ and C_1 can be included in an sl_2 subalgebra containing I_0. The grade of Λ is one with respect to the grading operator $d_{m,I_0} = m\lambda\frac{d}{d\lambda} + \mathrm{ad}I_0$.

In the appendix it will be observed that the same relationship is valid between arbitrary *regular primitive* conjugacy classes in the Weyl group and certain sl_2 embeddings for *arbitrary* simple Lie algebras. For a non-primitive regular conjugacy class $[w]$ in the Weyl group of an exceptional simple Lie algebra different from G_2, in some cases the order of $w \in [w]$ is smaller than the largest spin plus one with respect to the sl_2 associated to $[w]$.

Applying the above group theoretic results, we provided a link between the generalized KdV hierarchies and $\mathcal{W}$-algebras and made a step towards obtaining a more concrete description of the KdV systems. In particular, we derived Gelfand-Dickey type Lax operators for the KdV systems associated to grade one regular elements from such Heisenberg subalgebras that are contained in a regular reductive subalgebra of a classical Lie algebra $\mathcal{G}$ comprising A and C type simple factors. In these cases the generalized KdV systems turned out to be discrete reductions of systems related to gl_n having Lax operators of the form given in (1.3) and (1.4).

The most interesting non-principal case occurring for the classical Lie algebras appears to be given by the regular primitive conjugacy class $(\bar{p}, \bar{p}) \subset \mathbf{W}(D_{2p})$, since the corresponding Heisenberg subalgebra is not contained in a regular reductive subalgebra. It is an intriguing question whether the generalized KdV system associated to a grade one regular element with the aid of Drinfeld-Sokolov reduction admits a Gelfand-Dickey type pseudo-differential operator model in this case or not. Such a model is usually not hard to find using the elimination procedure, but for $(\bar{p}, \bar{p}) \subset \mathbf{W}(D_{2p})$ we did not succeed until now. The corresponding non-abelian affine Toda system presented in Section 6 would also deserve further investigation.

In this study we used the interplay between the homogeneous grading and the grading given by d_{m,I_0} to define the constraints on the first order differential operator $\mathcal{L} = \partial + j + \Lambda$ containing the dynamical variables. It is known [1, 7, 11, 12] that there are more general possibilities: *i)* the d_{m,I_0} grading can be replaced by an arbitrary grading in which Λ has definite grade; *ii)* the homogeneous grading can be replaced by another standard grading (associated to an appropriate vertex of the extended Dynkin diagram) or a grading interpolating between a standard grading and the grading in which Λ has definite grade. See also the remark at the end of Section 2. It would be interesting to further explore these more general possibilities for

obtaining KdV and partially modified KdV systems, which are related to the same basic set of modified KdV systems by different Miura maps [1, 7, 11, 12].

We wish to remark that in some cases the partially modified systems correspond to partial factorizations of a Lax operator that can be factorized into factors of order one, not unlike the example when say a fourth order KdV operator L is partially factorized into operators of order two according to $L = (\partial + \theta_1)(\partial + \theta_2)(\partial + \theta_3)(\partial + \theta_4) = L_1 L_2$ with $L_1 = (\partial + \theta_1)(\partial + \theta_2)$ and $L_2 = (\partial + \theta_3)(\partial + \theta_4)$.

We have restricted our attention to regular elements of *minimal* grade. According to an argument in [12, 13], the systems associated to regular elements of higher grade in a certain sense should not be new, although the Hamiltonian aspect of this claim is not well understood.

Perhaps the most serious limitation of the present work is that we excluded "type II" systems, that is systems associated to graded *non-regular* semisimple elements of $\ell(\mathcal{G})$, from the outset. It is an important open problem to classify the gradings that admit graded semisimple elements for which Drinfeld-Sokolov reduction is possible in the sense that polynomial "DS gauges" exist. Some results on type II systems including interesting examples can be found in [45, 46, 11, 15, 18]. In particular, it was recently shown in [15] that the phase space of the partially modified systems contains standard $\mathcal{W}$-algebras coupled together by the dynamics in both type I and type II cases subject to a certain non-degeneracy condition.

It is worth noting that the regular conjugacy classes in the groups obtained as extensions of the Weyl groups by diagram automorphisms have been also classified in [27], which is relevant for constructing generalized KdV and affine Toda systems based on the twisted loop algebras.

To conclude, we think the general framework of the Drinfeld-Sokolov approach is now reasonably clear but further work would be needed to fully classify the integrable hierarchies that can be obtained from this approach. For instance, it would be of some interest to further explore the KdV systems that may be defined using arbitrary grade one regular semisimple elements and arbitrary standard gradings and type II systems would also deserve closer attention.

Acknowledgements. We wish to thank D. Olive, P. Sorba, F. Toppan and J. Underwood for discussions. We are grateful to I.R. McIntosh for making available ref. [11] and to a referee for drawing our attention to ref. [27]. We are indebted to A. Honecker, I. Marshall and I. Tsutsui for comments and suggestions concerning the manuscript.

A Canonical sl_2 for any regular primitive conjugacy class

The purpose of this appendix is to present a property of the regular primitive conjugacy classes in $\mathbf{W}(\mathcal{G})$ that generalizes the celebrated relationship [35] between the Coxeter class and the principal sl_2 subalgebra of $\mathcal{G}$. We find this relationship by collecting known results in the literature. A larger set of regular conjugacy classes enjoying the attractive features of this relationship (properties 1–7 below) will be also pointed out.

Let $\mathcal{G}$ be an arbitrary simple Lie algebra. The primitive (semi-Coxeter) conjugacy classes in $\mathbf{W}(\mathcal{G})$ are the building blocks of the general conjugacy classes [26] and the *regular primitive* conjugacy classes are the building blocks of the general regular conjugacy classes. The Coxeter class, whose Carter diagram [26] is the Dynkin diagram of $\mathcal{G}$, is the only primitive conjugacy class for the algebras of A, B, C and G_2 type. The other primitive conjugacy classes can be uniquely labelled by the Carter diagrams $D_l(a_i)$ for $i = 1, \ldots, [l/2] - 1$, $F_4(a_1)$, $E_6(a_i)$ for $i = 1, 2$, $E_7(a_i)$ for $i = 1, \ldots, 4$ and $E_8(a_i)$ for $i = 1, \ldots, 8$. The Coxeter class is always regular. Comparing the characteristic polynomials of the primitive conjugacy classes given in [26] with those of the regular conjugacy classes given in [27], it can be seen that the other *regular primitive* conjugacy classes are $D_{2k}(a_{k-1}) \sim (\bar{k}, \bar{k})$ in $\mathbf{W}(D_{2k})$ for $k = 2, 3, \ldots$, and

$$F_4(a_1), \quad E_6(a_1),\ E_6(a_2), \quad E_7(a_1),\ E_7(a_4), \quad E_8(a_i) \quad \text{for} \quad i = 1, 2, 3, 5, 6, 8. \tag{A.1}$$

Putting together results of [35, 27, 39, 41], we notice the validity of the following statement.

Theorem A. *Let $[w] \subset \mathbf{W}(\mathcal{G})$ be an arbitrary regular primitive conjugacy class of order N. Then there exists a lift $\hat{w}$ of $w \in [w]$ given by an inner automorphism of $\mathcal{G}$ that has the form*

$$\hat{w} = \exp\left(2i\pi \mathrm{ad} I_0 / N\right), \tag{A.2}$$

where I_0 is the semisimple element of an sl_2 subalgebra of $\mathcal{G}$, $[I_0, I_\pm] = \pm I_\pm$, $[I_+, I_-] = 2I_0$, such that

1. *The largest eigenvalue of $\mathrm{ad} I_0$ equals $(N-1)$.*
2. *There are no singlets in the sl_2 decomposition of $\mathcal{G}$.*
3. *Only integral eigenvalues of $\mathrm{ad} I_0$ occur.*

Verification. The case of the Coxeter class is due to Kostant [35]. The characteristic diagrams [34] of the sl_2 embeddings corresponding[5] to the conjugacy classes

$$E_6(a_1), \quad E_7(a_1), \quad E_8(a_1),\ E_8(a_2),\ E_8(a_5) \tag{A.3}$$

are given in Table 11 of Ref. [27], where the statement is proved concerning these cases. (See also remarks iii) and vii) below.) In the algebras of E type, the “shift vector” $\gamma_s \in \mathcal{G}$ defining a so called canonical lift of a representative $w \in [w]$ was determined by Bouwknegt [41] for all conjugacy classes $[w] \subset \mathbf{W}(\mathcal{G})$. For the definition and for the rather complex method whereby γ_s was obtained, see [41]. In the case of the primitive conjugacy classes this canonical lift takes the form $\hat{w} = \exp\left(2i\pi \mathrm{ad}\gamma_s / N\right)$. Comparing the tables of [41] with the tables of Dynkin [34],

[5] In [27] there is a misprint in the diagram of the sl_2 with Dynkin index 280 that corresponds to $E_8(a_5)$.

one can verify that $\gamma_s \in \mathcal{G}$ coincides with the defining vector of an sl_2 embedding *if and only if* the conjugacy class is regular. The sl_2 embeddings corresponding to the conjugacy classes

$$E_6(a_2), \quad E_7(a_4), \quad E_8(a_3),\ E_8(a_6),\ E_8(a_8) \tag{A.4}$$

are in this way identified as those with Dynkin index [34]

$$36, \quad 39, \quad 184,\ 120,\ 40, \tag{A.5}$$

respectively. Properties 1, 2, 3 can be checked. In the $D_{2k}(a_{k-1})$ cases the lift satisfying the statement of the theorem was determined in [39], as we have discussed in Subsection 4.2 using the alternative parametrization $D_{2k}(a_{k-1}) \sim (\bar{k}, \bar{k})$. The remaining $F_4(a_1)$ case results from the $E_6(a_2)$ case by applying the canonical diagram automorphism τ of E_6, whose fixed point set is F_4. This is similar to an appropriate representative of the Coxeter class of E_6 reducing to a representative of the Coxeter class of F_4 on the fixed point set of τ, which is well-known. In fact, $E_6(a_2)$ and $F_4(a_1)$ can be represented by the squares of the respective Coxeter elements. The sl_2 embedding associated to $E_6(a_2)$ by the theorem is the principal sl_2 in the regular subalgebra $(A_5+A_1) \subset E_6$, which is the same as the principal sl_2 in the regular subalgebra $(C_3+A_1) \subset F_4$. Using also Lemma 9.5 of Springer [27], we can conclude that the latter sl_2 subalgebra of F_4, having Dynkin index 36, satisfies the statement of the theorem for $F_4(a_1)$. *Q.E.D.*

Any representative of a regular primitive conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$ of order N has [27] a regular semisimple eigenvector associated to the eigenvalue $\omega_N := \exp(2i\pi/N)$. For the lift $\hat{w}$ given in the theorem, any semisimple eigenvector H of eigenvalue ω_N has the form

$$H = C_1 + C_{-(N-1)} \quad \text{with} \quad C_k \neq 0, \quad [I_0, C_k] = kC_k. \tag{A.6}$$

We have the following consequence of the theorem.

Corollary A. *Let $\hat{w}$ be the lift of a regular primitive conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$ given in the theorem and H in (A.6) be a regular semisimple eigenvector of $\hat{w}$ with eigenvalue ω_N. Let $\mathcal{H}_H \subset \mathcal{G}$ be the Cartan subalgebra defined as the centralizer of H. Then*

4. The restriction of $\hat{w}$ to $\mathcal{H}_H$ acts as a representative of the conjugacy class $[w] \subset \mathbf{W}(\mathcal{G})$.
5. I_0 and C_1 can be completed to an sl_2 subalgebra of $\mathcal{G}$.

Proof. Since $\hat{w}$ maps $\mathcal{H}_H$ to itself it defines a representative of a conjugacy class in $\mathbf{W}(\mathcal{G})$. This conjugacy class is obviously regular and has order N. Property 4 follows since there can be only one regular conjugacy classes of a given order [27]. To show property 5, notice that $\dim \mathcal{G}^{I_0}_{-1} = \dim \mathcal{G}^{I_0}_0$ by property 2 in the theorem. Further notice that

$$\mathrm{Ker}\,(\mathrm{ad}C_1) \cap \mathcal{G}^{I_0}_{<0} = \{0\} \tag{A.7}$$

by property 1 and by the assumption that H in (A.6) is regular semisimple. *Q.E.D.*

Let I_0, $I_\pm$ be the sl_2 subalgebra given in the theorem and $C_{-(N-1)}$ some element of $\mathcal{G}^{I_0}_{-(N-1)}$. Note that for I_0 given $I_\pm$ are not unique. Springer [27] has also shown the following:

6. If $(I_+ + C_{-(N-1)})$ is semisimple then it is regular semisimple.
7. There exists $C_{-(N-1)}$ such that $(I_+ + C_{-(N-1)})$ is regular semisimple.

We wish to make some further remarks on the "canonical correspondence" between sl_2 embeddings and regular primitive conjugacy classes established above.

i) The shift vector defining the canonical lift [41, 39] of a primitive conjugacy class in $\mathbf{W}(\mathcal{G})$ determines an sl_2 embedding *only* if the conjugacy class is regular.

ii) The sl_2 corresponding to a regular primitive (semi-Coxeter) conjugacy class is *not* always a singular (semi-principal) sl_2.

iii) The principal sl_2 and the sl_2 subalgebras corresponding to the conjugacy classes in (A.3) satisfy [27] in addition to properties 2, 3 also the property that there occurs only one triplet in the sl_2 decomposition of $\mathcal{G}$. There exists only one additional sl_2 embedding with these properties, corresponding to the regular embedding $B_4 \subset F_4$. The multiplicity of the largest spin sl_2 multiplet in $\mathcal{G}$ is also one in these cases.

iv) Relation (A.2) alone would *not* determine uniquely the conjugacy class of the sl_2 generator I_0 (think of non-conjugate powers of a Coxeter element). It may be checked that (A.2) together with property 1 does so.

v) The shift vector determined in [41] for all the conjugacy classes in $\mathbf{W}(E_{6,7,8})$ associates an sl_2 embedding to every regular conjugacy class. Property 1 is *not* always satisfied, but the weaker requirement $\mathcal{G}_1^{\hat{w}} = \mathcal{G}_0^{I_0}$ holds for the eigenspaces of $\hat{w}$ and $\mathrm{ad} I_0$ with the respective eigenvalues. If property 1 is not satisfied then it might be necessary to modify the definition of the Drinfeld-Sokolov reduction used in Section 5, because relation (5.2), $[C_-, \mathcal{G}^{I_0}_{<0}] = \{0\}$, is then *not* guaranteed to hold for the grade one regular semisimple element $\Lambda = (I_+ + \lambda C_-)$.

vi) There exist a few other sl_2 embeddings and non-primitive regular conjugacy classes in the Weyl group of a simple Lie algebra $\mathcal{G}$ for which *all* of the above presented equations and properties 1–7 hold true as well. These conjugacy classes in $\mathbf{W}(\mathcal{G})$ are given by the following Carter diagrams:

$$D_{2k}(a_{k-1}) \in \mathbf{W}(B_{2k}) \quad \text{for} \quad k \geq 1, \quad A_2 \in \mathbf{W}(G_2), \quad B_4 \in \mathbf{W}(F_4), \quad D_4(a_1) \in \mathbf{W}(F_4), \tag{A.8}$$

where for $k = 1$ we use the definition $D_2(a_0) := (\bar{1}, \bar{1})$. The corresponding sl_2 is obtained by taking the semi-principal or principal sl_2 embedding in the respective regular simple subalgebras of maximal rank,

$$D_{2k}(a_{k-1}) \subset B_{2k} \quad \text{for} \quad k \geq 1, \qquad A_2 \subset G_2, \qquad B_4 \subset F_4, \qquad D_4(a_1) \subset F_4, \tag{A.9}$$

where $D_{2k}(a_{k-1})$ denotes the semi-principal sl_2 subalgebra in D_{2k} described in Subsection 4.2. For the alert reader, we note that $D_4(a_1) \subset F_4$ is the sl_2 of Dynkin index 12, although this labelling of it is missing in the table of [34].

vii) Springer [27] studied the correspondence between sl_2 embeddings and regular conjugacy classes in the Weyl group using in addition to (A.2) and properties 1, 2, 3 the assumption that there exists a regular semisimple eigenvector of $\hat{w}$ given by (A.2) of the form in (A.6). It can be checked that the sl_2 subalgebras corresponding to the regular primitive conjugacy classes together with those in (A.8) yield the *exhaustive set* for which these assumptions are satisfied. In [27] the strong additional assumption that the decomposition of $\mathcal{G}$ under the sl_2 contains only one triplet was used to ensure the existence of a regular semisimple eigenvector.

In the above we have described a canonical correspondence between the regular primitive conjugacy classes in the Weyl group and certain associated sl_2 embeddings. The correspondence

enjoys a set of attractive properties, which are shared by certain other regular non-primitive conjugacy classes, given in (A.8), and corresponding sl_2 embeddings. Some further nice properties valid in these cases can be found in [27]. This correspondence enhances our understanding of the classification of integrable hierarchies associated to regular conjugacy classes in the Weyl group and could be further exploited in more detailed studies of these systems.

References

[1] V.G. Drinfeld and V.V. Sokolov, Sov. Math. Dokl. **23** (1981) 457; J. Sov. Math. **30** (1985) 1975.

[2] I.M. Gelfand and L.A. Dickey, Funct. Anal. Appl. **10:4** (1976) 13; Funct. Anal. Appl. **11:2** (1977) 93.

[3] L.A. Dickey, *Soliton Equations and Hamiltonian Systems*, Adv. Ser. Math. Phys., Vol. 12, World Scientific, Singapore, 1991.

[4] M. Adler, Invent. Math. **50:3** (1979) 219.

[5] A.G. Reyman and M.A. Semenov-Tian-Shansky, Funct. Anal. Appl. **14** (1980) 146.

[6] V.G. Kac, *Infinite Dimensional Lie Algebras*, Cambridge University Press, Cambridge, 1985.

[7] G.W. Wilson, Ergod. Th. and Dynam. Sys. **1** (1981) 361.

[8] A.N. Leznov and M.V. Saveliev, Commun. Math. Phys. **89** (1983) 59.

[9] D. Olive and N. Turok, Nucl. Phys. **B257** (1985) 277.

[10] J.W.R. Underwood, London preprint IC/TP/92-93/30, hep-th/9304156; *Toda Theories as Model Integrable Systems*, PhD Thesis, Dept. Phys., Imperial College, London, 1993.

[11] I.R. McIntosh, *An Algebraic Study of Zero Curvature Equations*, PhD Thesis, Dept. Math., Imperial College, London, 1988.

[12] M.F. de Groot, T.J. Hollowood and J.L. Miramontes, Commun. Math. Phys. **145** (1992) 57.

[13] N.J. Burroughs, M.F. de Groot, T.J. Hollowood and J.L. Miramontes, Commun. Math. Phys. **153** (1993) 187; Phys. Lett. **B277** (1992) 89.

[14] T.J. Hollowood and J.L. Miramontes, Commun. Math. Phys. **157** (1993) 99.

[15] C.R. Fernández-Pousa, M.V. Gallas, J.L. Miramontes and J.S. Guillén, Santiago de Compostela preprint, US-FT/13-94, hep-th/9409016.

[16] V.G. Kac and D.H. Peterson, pp. 276-298 in: Proc. of Symposium on Anomalies, Geometry and Topology, W.A. Bardeen and A.R. White (eds.), World Scientific, Singapore, 1985.

[17] L. Fehér, J. Harnad and I. Marshall, Commun. Math. Phys. **154** (1993) 181.

[18] L. Fehér and I. Marshall, Swansea preprint, SWAT-95-61, hep-th/9503217.

[19] F. ten Kroode and J. van de Leur, Commun. Math. Phys. **137** (1991) 67.

[20] Yi Cheng, J. Math. Phys. **33** (1992) 3774.

[21] W. Oevel and W. Strampp, Commun. Math. Phys. **157** (1993) 51.

[22] A. Deckmyn, Phys. Lett. **B298** (1993) 318.

[23] L.A. Dickey, Oklahoma preprint, hep-th/9407038.

[24] L. Bonora, Q.P. Liu and C.S. Xiong, Bonn preprint, BONN-TH-94-17, hep-th/9408035.

[25] H. Aratyn, J.F. Gomes and A.H. Zimerman, Chicago preprint, UICHEP-TH/93-10, hep-th/9408104.

[26] R.W. Carter, Comp. Math. **25** (1972) 1.

[27] T.A. Springer, Inventiones math. **25** (1974) 159.

[28] A.B. Zamolodchikov, Theor. Math. Phys. **65** (1985) 1205.

[29] M. Douglas, Phys. Lett. **B238** (1990) 176.

[30] F.A. Bais, T. Tjin and P. van Driel, Nucl. Phys. **B357** (1991) 632.

[31] L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222** (1992) 1.

[32] P. Bouwknegt and K. Schoutens, Phys. Rep. **223** (1993) 183.

[33] A. Morozov, Amsterdam preprint, IFTA-93-10 (1993).

[34] E.B. Dynkin, Amer. Math. Soc. Transl. **6** [**2**] (1957) 111.

[35] B. Kostant, Am. J. Math. **81** (1959) 973.

[36] A. Pressly and G. Segal, *Loop Groups*, Oxford University Press, 1986.

[37] F. ten Kroode, *Affine Lie Algebras and Integrable Systems*, PhD Thesis, University of Amsterdam, 1988.

[38] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.

[39] F. ten Kroode and J. van de Leur, Comm. in Algebra **20** (1992) 3119.

[40] R.W. Carter and G.B. Elkington, Journal of Algebra **20** (1972) 350.

[41] P. Bouwknegt, J. Math. Phys. **30** (1989) 571-584.

[42] N. Jacobson, *Lie Algebras*, Wiley-Interscience, New York, 1962.

[43] A.G. Reyman and M.A. Semenov-Tian-Shansky, Phys. Lett. **A130** (1988) 456.

[44] J. Balog, L. Fehér, L. O'Raifeartaigh, P. Forgács and A. Wipf, Ann. Phys. (N.Y.) **203** (1990) 76.

[45] A.P. Fordy and P.P. Kulish, Commun. Math. Phys. **89** (1983) 427.

[46] I.R. McIntosh, J. Math. Phys. **34** (1993) 5159.